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Nonlinear kinetic turbulence theory

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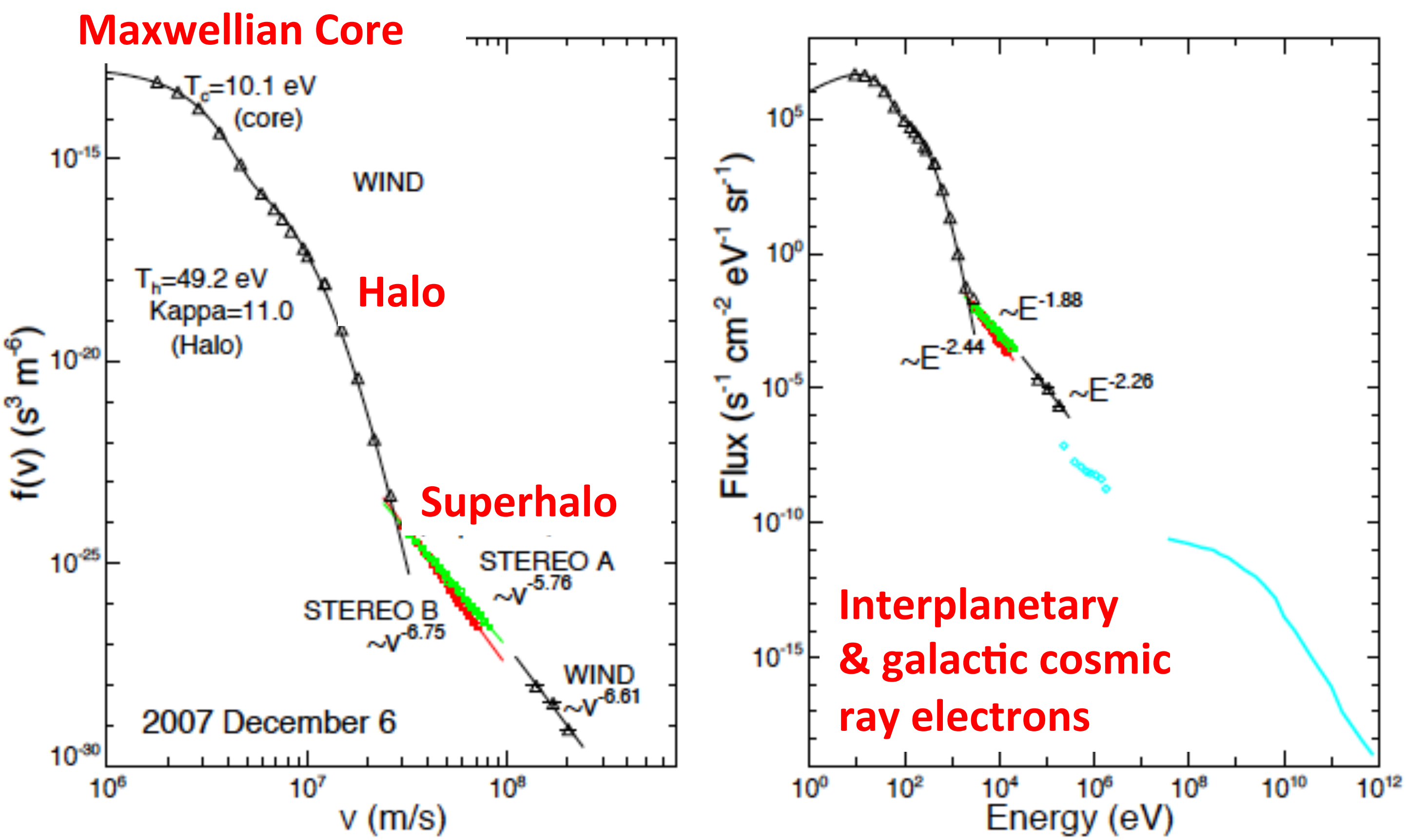
	FLUID THEORIES	REDUCED KINETIC THEORIES	KINETIC THEORIES
UNMAGNETIZED	• Strong turbulence (*à la* Zakharov)		• **Weak turbulence for unmagnetized plasmas (Tsytovich, Melrose, Sitenko, Davidson, etc.). Most mature kinetic plasma turbulence theory** • Renormalized (strong turbulence *à la Kadomtsev,* Dupree, etc.)
MAGNETIZED	• MHD (Ideal, Resistive, Hall, etc). Well-developed mature field. • Drift-wave turb. (Hasegawa-Mima)	• Drift Kinetic • Gyrokinetic (applicable for low-frequency regime and for strongly magnetized plasmas)	• Weak turbulence for magnetized plasmas (incomplete, but some preliminary works include those by Pustovalov & Silin, Porkolab, etc.)

Part 1. Electrostatic Problem

SOLAR WIND ELECTRON DISTRIBUTION AT 1 AU



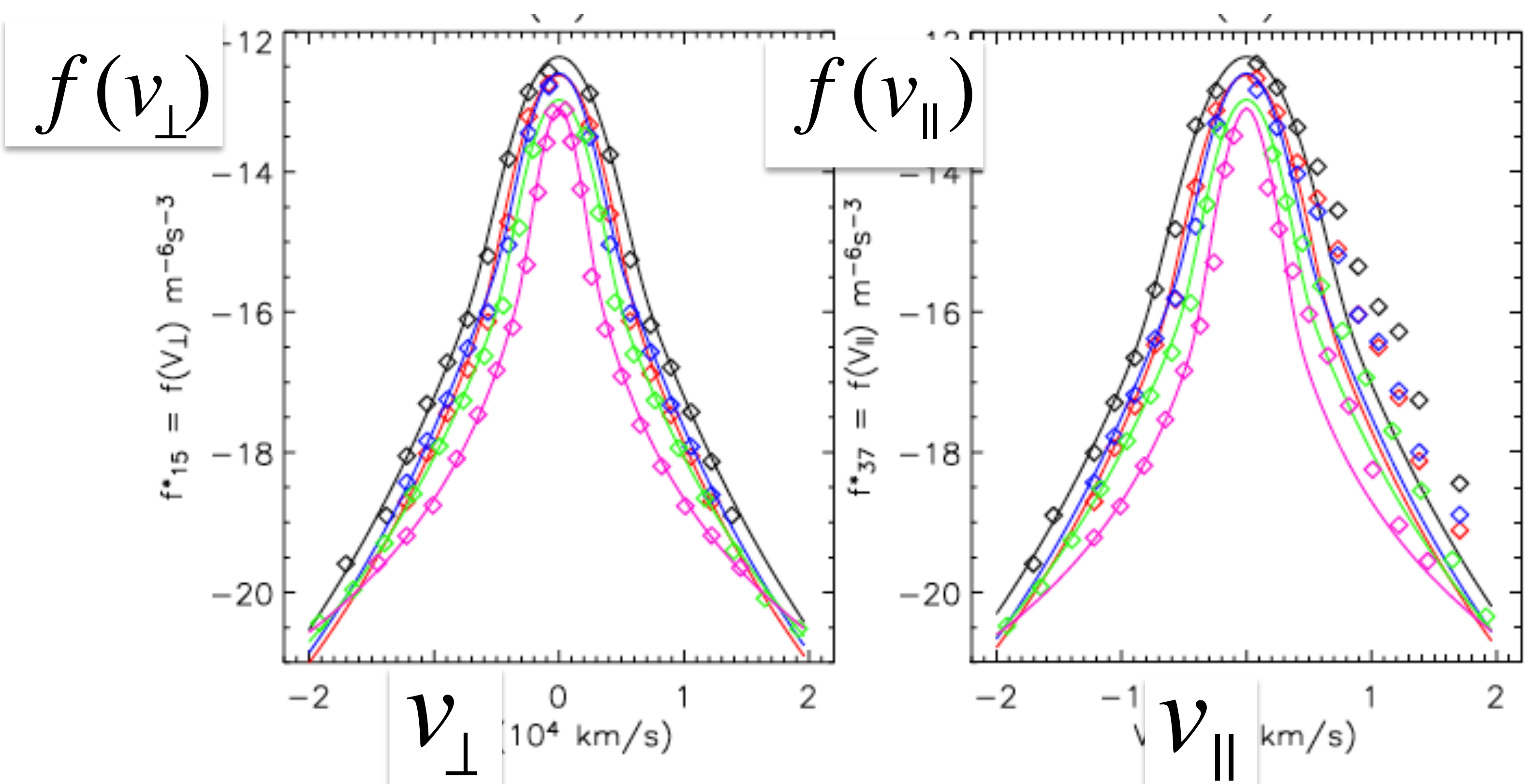
Wang et al., *ApJ Lett.* (2012)

Maksimovic et al. *JGR*, *110*, A09104 (2005)

$$f = n_c f_c + n_h f_h,$$

(bi) kappa model

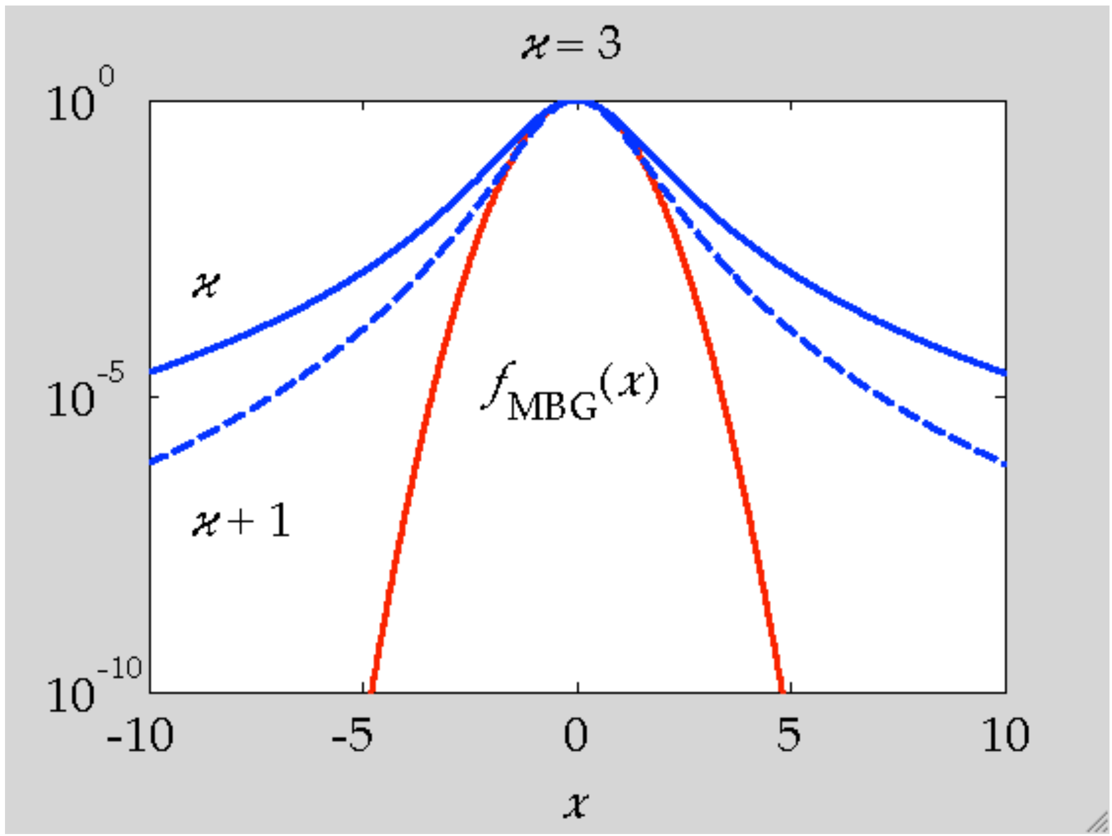
$$f_c \propto \exp\left(-\frac{v_\perp^2}{\alpha_{\perp c}^2} - \frac{v_\parallel^2}{\alpha_{\parallel c}^2}\right), \qquad f_h \propto \frac{1}{\left(1 + \frac{v_\perp^2}{\kappa\alpha_{\perp h}^2} + \frac{v_\parallel^2}{\kappa\alpha_{\parallel h}^2}\right)^{-\kappa}}$$



$$f_{Maxwell-Boltzmann-Gauss} \propto e^{-x^2}$$

$$f_{kappa} \propto \frac{1}{(1+x^2/\kappa)^{\kappa}} \quad or \quad \frac{1}{(1+x^2/\kappa)^{\kappa+1}}$$

$$e^{-x} = \lim_{\kappa \to \infty} \left(1 + \frac{x}{\kappa}\right)^{-\kappa}$$



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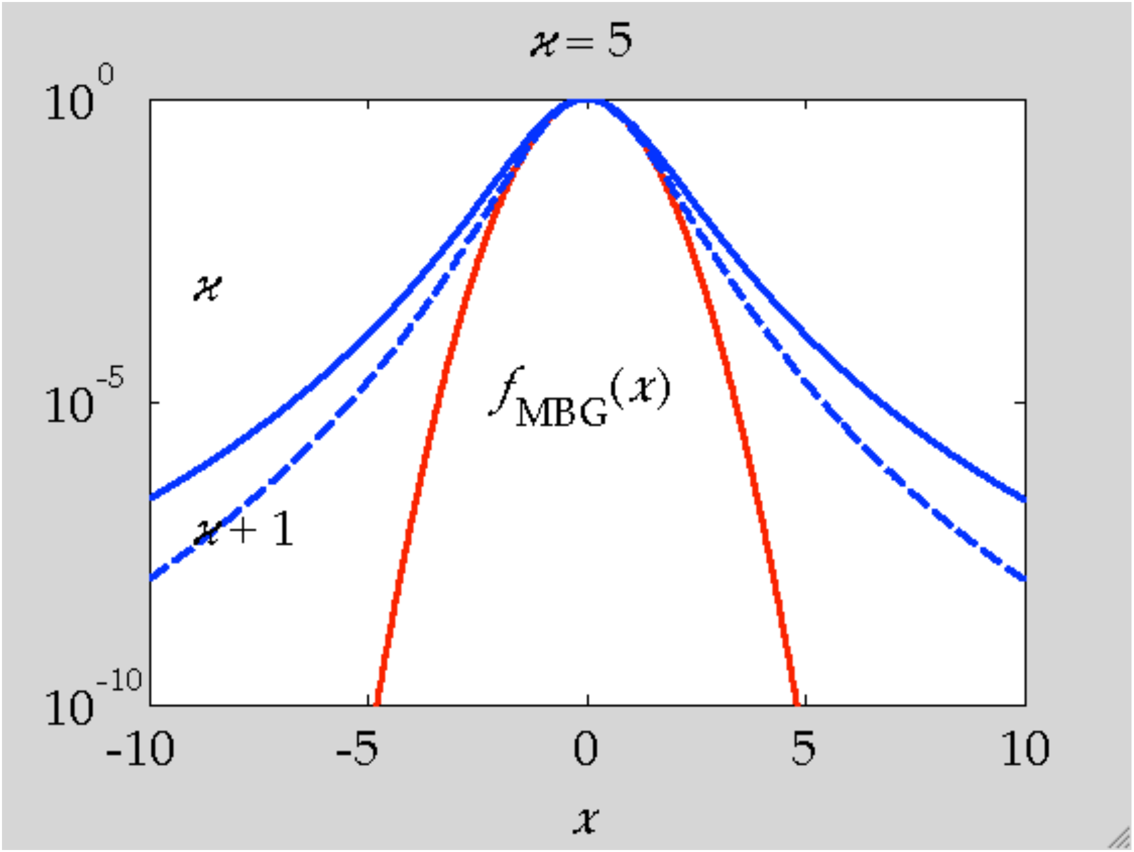

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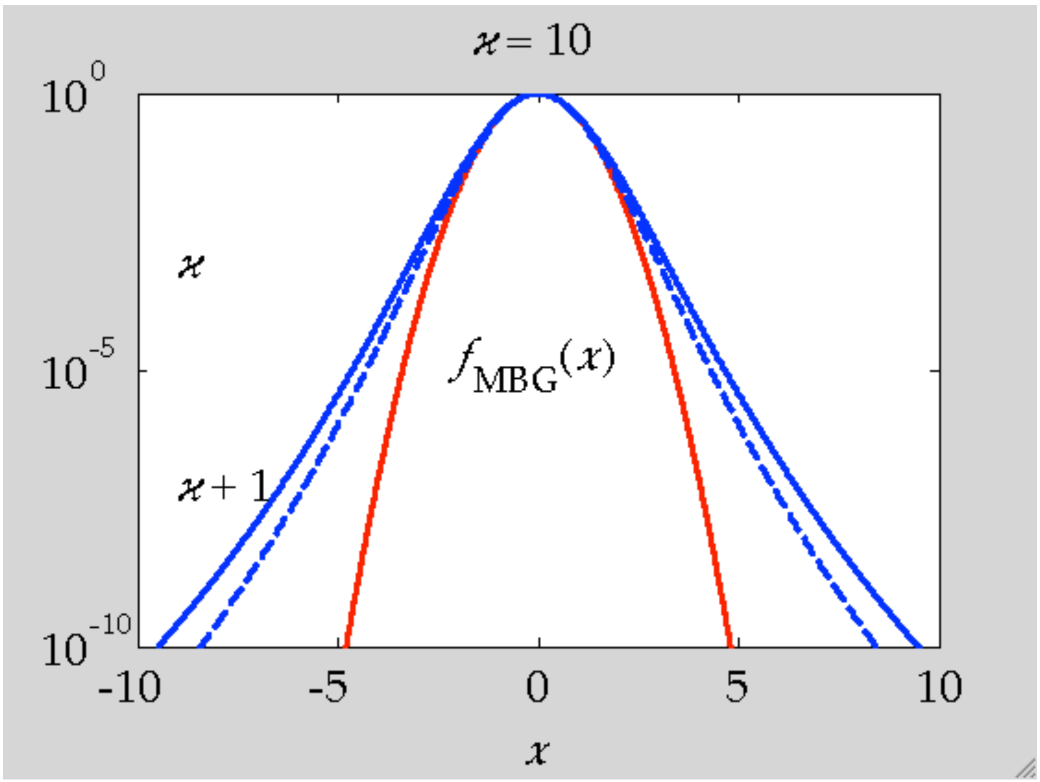
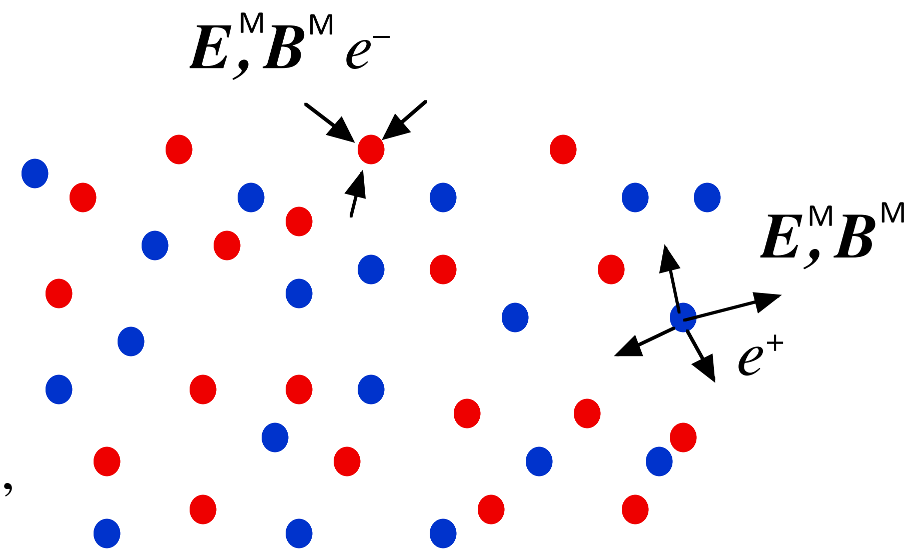


Theories of "Kappa" Distribution

- 1. Collisional or Stochastic Transport Model (Scudder, 1990s; Schwadron, 2010)
- 2. Non-Extensive Thermodynamics (Tsallis, 1980s; Treumann, 2000s; Leubner, 2000s; Livadiotis, 2000s)
- 3. Steady-State Plasma Turbulence (Yoon, 2014)

Nonlinear Plasma Interaction



$$\frac{d\mathbf{r}_i^a(t)}{dt} = \mathbf{v}_i^a(t),$$

$$\frac{d\mathbf{v}_i^a(t)}{dt} = e_a\mathbf{E}[\mathbf{r}_i^a(t),t] + \frac{e_a}{c}\mathbf{v}_i^a(t)\times\mathbf{B}[\mathbf{r}_i^a(t),t],$$

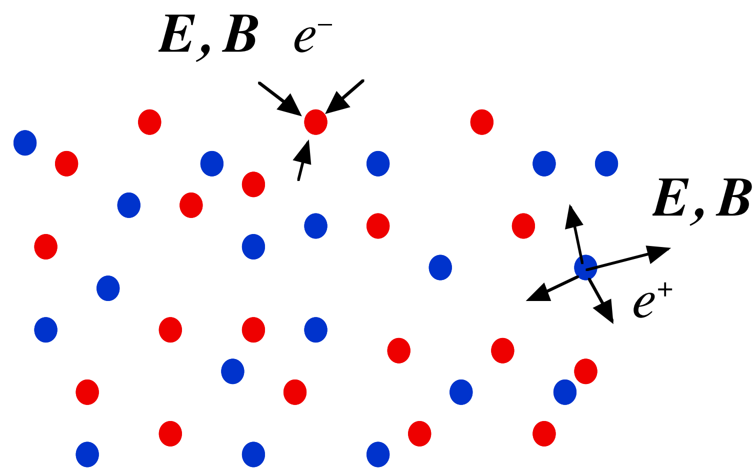
$$\nabla\times\mathbf{E}(\mathbf{r},t) + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t) = 0,$$

$$\nabla\bullet\mathbf{B}(\mathbf{r},t) = 0,$$

$$\nabla\bullet\mathbf{E}(\mathbf{r},t) = 4\pi\sum_a\sum_{i=1}^{N}\delta[\mathbf{r}-\mathbf{r}_i^a(t)],$$

$$\nabla\times\mathbf{B}(\mathbf{r},t) - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t) = \frac{4\pi}{c}\sum_a e_a\sum_{i=1}^{N}\mathbf{v}_i^a(t)\delta[\mathbf{r}-\mathbf{r}_i^a(t)].$$

Klimontovich function



$$N_a(\mathbf{r},\mathbf{v},t) = \sum_{i=1}^{N} \delta[\mathbf{r} - \mathbf{r}_i^a(t)]\delta[\mathbf{v} - \mathbf{v}_i^a(t)],$$

$$\left[\frac{\partial}{\partial t} + \mathbf{v} \bullet \nabla + \frac{e_a}{m_a}\left(\mathbf{E}(\mathbf{r},t) + \frac{\mathbf{v}}{c} \times \mathbf{B}(\mathbf{r},t)\right) \bullet \frac{\partial}{\partial \mathbf{v}}\right] N_a(\mathbf{r},\mathbf{v},t) = 0,$$

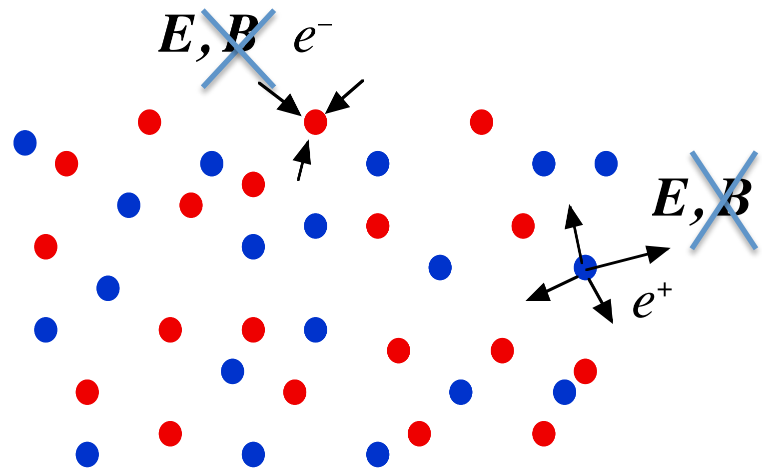
$$\nabla \times \mathbf{E}(\mathbf{r},t) + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t) = 0,$$

$$\nabla \bullet \mathbf{B}(\mathbf{r},t) = 0,$$

$$\nabla \bullet \mathbf{E}(\mathbf{r},t) = 4\pi \sum_a \int d\mathbf{v} N_a(\mathbf{r},\mathbf{v},t),$$

$$\nabla \times \mathbf{B}(\mathbf{r},t) - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t) = \frac{4\pi}{c}\sum_a e_a \int d\mathbf{v}\mathbf{v} N_a(\mathbf{r},\mathbf{v},t).$$

Electrostatic approximation



$$N_a(\mathbf{r},\mathbf{v},t) = \sum_{i=1}^{N} \delta[\mathbf{r}-\mathbf{r}_i^a(t)]\delta[\mathbf{v}-\mathbf{v}_i^a(t)],$$

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\bullet\nabla + \frac{e_a}{m_a}\mathbf{E}(\mathbf{r},t)\bullet\frac{\partial}{\partial \mathbf{v}}\right]N_a(\mathbf{r},\mathbf{v},t) = 0,$$

$$\nabla\bullet\mathbf{E}(\mathbf{r},t) = 4\pi\sum_a \int d\mathbf{v}N_a(\mathbf{r},\mathbf{v},t).$$

Separation into average and fluctuation

$$< N_a(\mathbf{r},\mathbf{v},t) >= f_a(\mathbf{r},\mathbf{v},t)$$

$$N_a(\mathbf{r},\mathbf{v},t) = f_a(\mathbf{r},\mathbf{v},t) + \delta N_a(\mathbf{r},\mathbf{v},t)$$

$$\mathbf{E}(\mathbf{r},t) = \delta\mathbf{E}(\mathbf{r},t)$$

$$< N_a(\mathbf{r},\mathbf{v},t) >= f_a(\mathbf{r},\mathbf{v},t)$$

$$N_a(\mathbf{r},\mathbf{v},t) = f_a(\mathbf{r},\mathbf{v},t) + \delta N_a(\mathbf{r},\mathbf{v},t)$$

$$\mathbf{E}(\mathbf{r},t) = -\nabla \delta\phi(\mathbf{r},t)$$

Particle kinetic equation

$$\frac{\partial f_a}{\partial t} = \frac{-ie_a}{m_a}\int d\mathbf{k}\mathbf{k} \bullet \frac{\partial}{\partial \mathbf{v}} < \delta\phi_{-\mathbf{k}}\delta N_{\mathbf{k}}^a >,$$

Fluctuation Equation

$$\left(\frac{\partial}{\partial t} + i\mathbf{k} \bullet \mathbf{v}\right)[\delta N_{\mathbf{k}} - \delta N_{\mathbf{k}}^0] = \frac{ie_a}{m_a}\delta\phi_{\mathbf{k}}\mathbf{k} \bullet \frac{\partial f_a}{\partial \mathbf{v}}$$

$$+\frac{ie_a}{m_a}\int d\mathbf{k}'\mathbf{k}' \bullet \frac{\partial}{\partial \mathbf{v}}[\delta\phi_{\mathbf{k}'}\delta N_{\mathbf{k}-\mathbf{k}'}^a - < \delta\phi_{\mathbf{k}'}\delta N_{\mathbf{k}-\mathbf{k}'}^a >],$$

$$\delta\phi_{\mathbf{k}} = \sum_a \frac{4\pi e_a}{k^2}\int d\mathbf{v}\delta N_{\mathbf{k}}^a .$$

Field equation

Brief Overview of Weak Turbulence Theory

$$\left[\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}+\frac{e_a E}{m_a}\frac{\partial}{\partial v}\right]f_a=0,\qquad \frac{\partial E}{\partial x}=4\pi n\sum_a e_a\int dv\, f_a$$

$$f_a=<N_a>+\delta f_a=F_a+\delta f_a,\qquad E=\delta E$$

$$\left(\frac{\partial}{\partial t}+\frac{e_a}{m_a}\delta E\frac{\partial}{\partial v}\right)F_a+\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}+\frac{e_a}{m_a}\delta E\frac{\partial}{\partial v}\right)\delta f_a=0,$$

$$\frac{\partial}{\partial x}\delta E=4\pi n\sum_a e_a\int dv\,\delta f_a$$

Average over random phase: $$\frac{\partial F_a}{\partial t}=-\frac{e_a}{m_a}\frac{\partial}{\partial v}<\delta E\delta f_a>$$

Insert back to the original equation

$$\left(\frac{\partial}{\partial t}+\frac{e_a}{m_a}\delta E\frac{\partial}{\partial v}\right)F_a+\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}+\frac{e_a}{m_a}\delta E\frac{\partial}{\partial v}\right)\delta f_a=0$$

$$\frac{\partial F_a}{\partial t}=-\frac{e_a}{m_a}\frac{\partial}{\partial v}<\delta E\delta f_a>$$

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}\right)\delta f_a=-\frac{e_a}{m_a}\delta E\frac{\partial F_a}{\partial v}-\frac{e_a}{m_a}\frac{\partial}{\partial v}\left(\delta f_a\delta E-<\delta f_a\delta E>\right)$$

Two time scales (slow and fast)

$$\delta f_a(x,v,t)=\int dk\int d\omega\,\delta f^a_{k,\omega}(v,t)e^{ikx-i\omega t}$$

slow fast

$$\left(\omega-kv+i\frac{\partial}{\partial t}\right)\delta f^a_{k,\omega}=-\frac{ie_a}{m_a}\delta E_{k,\omega}\frac{\partial F_a}{\partial v}$$

$$-\frac{ie_a}{m_a}\frac{\partial}{\partial v}\int dk'\int d\omega'\left(\delta f^a_{k-k',\omega-\omega'}\delta E_{k',\omega'}-<^a_{k-k',\omega-\omega'}\delta E_{k',\omega'}>\right)$$

$$\left(\omega - kv + i\frac{\partial}{\partial t}\right)\delta f^{a}_{k,\omega} = -\frac{ie_a}{m_a}\delta E_{k,\omega}\frac{\partial F_a}{\partial v}$$

$$-\frac{ie_a}{m_a}\frac{\partial}{\partial v}\int dk'\int d\omega'\left(\delta f^{a}_{k-k',\omega-\omega'}\delta E_{k',\omega'} - <^{a}_{k-k',\omega-\omega'}\delta E_{k',\omega'}>\right)$$

- $\omega \rightarrow \omega + i\frac{\partial}{\partial t}$
- $K = (k,\omega), \quad g_K = -\frac{ie_a}{m_a}\frac{1}{\omega - kv + i0}\frac{\partial}{\partial v}$

$$f_K = g_K F E_K + \int dK' g_K (E_{K'} f_{K-K'} - < E_{K'} f_{K-K'} >)$$

- iterative solution: $f_K = f_K^{(1)} + f_K^{(2)} + \ldots$

- insert to Poisson eq: $E_K = -i\sum_a \frac{4\pi n e_a}{k}\int dv f_K$

$\varepsilon(K)$: linear dielectric response

$$0 = \left(1 + \sum_a \frac{4\pi n_a e_a i}{k} \int dv\, g_K F \right) E_K$$

$$+ \int dK' \sum_a \frac{4\pi n_a e_a i}{k} \int dv\, g_K g_{K-K'} F \left(E_{K'} E_{K-K'} - \langle E_{K'} E_{K-K'} \rangle \right)$$

$\chi^{(2)}(K' | K - K')$: (second-order) nonlinear response

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$$0 = \varepsilon(K) E_K + \int dK' \chi^{(2)}(K' | K - K') \left(E_{K'} E_{K-K'} - \langle E_{K'} E_{K-K'} \rangle \right)$$

$$0 = \varepsilon(K) \langle E_K E_{-K} \rangle + \int dK' \chi^{(2)}(K' | K - K') \langle E_{-K} E_{K'} E_{K-K'} \rangle$$

$$0 = \varepsilon(K) < E_K E_{-K} > + \int dK' \chi^{(2)}(K'|K-K') < E_{-K} E_{K'} E_{K-K'} >$$

At this point we reintroduce the slow-time derivative

$$\varepsilon(K) < E^2 >_{k,\omega} \rightarrow \varepsilon\left(k, \omega + i\frac{\partial}{\partial t}\right) < E^2 >_{k,\omega} \rightarrow \left(\varepsilon(K) + \frac{i}{2}\frac{\partial \varepsilon(K)}{\partial \omega}\frac{\partial}{\partial t}\right) < E^2 >_{k,\omega}$$

$$0 = \frac{i}{2}\frac{\partial \varepsilon(K)}{\partial \omega}\frac{\partial}{\partial t} < E^2 >_K + \mathrm{Re}\,\varepsilon(K) < E^2 >_K + i\,\mathrm{Im}\,\varepsilon(K) < E^2 >_K$$
$$+ \int dK' \chi^{(2)}(K'|K-K') < E_{-K} E_{K'} E_{K-K'} >$$

- $\mathrm{Re}\,\varepsilon(K) < E^2 >_K = 0$ Dispersion relation
- $\frac{\partial}{\partial t} < E^2 >_K = -\frac{2\,\mathrm{Im}\,\varepsilon(K)}{\partial \mathrm{Re}\,\varepsilon(K)/\partial \omega} < E^2 >_K$ Wave kinetic equation

$$+\mathrm{Im}\frac{2i}{\partial \mathrm{Re}\,\varepsilon(K)/\partial \omega}\int dK' \chi^{(2)}(K'|K-K') < E_{-K} E_{K'} E_{K-K'} >$$

Coupling $\frac{\partial F_a}{\partial t} = -\frac{e_a}{m_a}\frac{\partial}{\partial v} < \delta E \delta f_a >$ and $f_K = g_K F E_K$

we obtain the particle kinetic equation

$$\frac{\partial F_a}{\partial t} = -\frac{e_a}{m_a}\frac{\partial}{\partial v}\int dK g_K < E^2 >_K F$$

Summary: formal equations of weak turbulence theory

- $\mathrm{Re}\,\varepsilon(K) < E^2 >_K = 0$ — Dispersion relation
- $\frac{\partial F_a}{\partial t} = -\frac{e_a}{m_a}\frac{\partial}{\partial v}\int dK g_K < E^2 >_K F$ — Particle kinetic equation
- $\frac{\partial}{\partial t} < E^2 >_K = -\frac{2\,\mathrm{Im}\,\varepsilon(K)}{\partial \mathrm{Re}\,\varepsilon(K)/\partial\omega} < E^2 >_K$ — Wave kinetic equation

$$+\mathrm{Im}\frac{2i}{\partial \mathrm{Re}\,\varepsilon(K)/\partial\omega}\int dK'\chi^{(2)}(K'|K-K') < E_{-K}E_{K'}E_{K-K'} >$$

Closure

$$\varepsilon(K)E_K = -\int dK'\chi^{(2)}(K'|K-K')\left(E_{K'}E_{K-K'} - \langle E_{K'}E_{K-K'}\rangle\right)$$

$$E_K = E_K^{(1)} + E_K^{(2)} + \cdots$$

$$\varepsilon(K)E_K^{(1)} + \varepsilon(K)E_K^{(2)} + \cdots = -\int dK'\chi^{(2)}(K'|K-K')\left(E_{K'}^{(1)}E_{K-K'}^{(1)} - \left\langle E_{K'}^{(1)}E_{K-K'}^{(1)}\right\rangle\right) + \cdots$$

$$\varepsilon(K)E_K^{(1)} = 0$$

$$E_K^{(2)} = -\frac{1}{\varepsilon(K)}\int dK'\chi^{(2)}(K'|K-K')\left(E_{K'}^{(1)}E_{K-K'}^{(1)} - \left\langle E_{K'}^{(1)}E_{K-K'}^{(1)}\right\rangle\right)$$

$$\left\langle E^{(1)}_{-K} E^{(1)}_{K'} E^{(1)}_{K-K'} \right\rangle = 0$$

$$\langle E_{-K} E_{K'} E_{K-K'} \rangle = \left\langle E^{(1)}_{-K} E^{(1)}_{K'} E^{(1)}_{K-K'} \right\rangle + \left\langle E^{(2)}_{-K} E^{(1)}_{K'} E^{(1)}_{K-K'} \right\rangle$$

$$+ \left\langle E^{(1)}_{-K} E^{(2)}_{K'} E^{(1)}_{K-K'} \right\rangle + \left\langle E^{(1)}_{-K} E^{(1)}_{K'} E^{(2)}_{K-K'} \right\rangle$$

$$\langle E_{-K} E_{K'} E_{K-K'} \rangle = \left\langle E^{(2)}_{-K} E^{(1)}_{K'} E^{(1)}_{K-K'} \right\rangle + \left\langle E^{(1)}_{-K} E^{(2)}_{K'} E^{(1)}_{K-K'} \right\rangle + \left\langle E^{(1)}_{-K} E^{(1)}_{K'} E^{(2)}_{K-K'} \right\rangle$$

$$\langle E_{-K} E_{K'} E_{K-K'} \rangle$$

$$= -\frac{1}{\varepsilon(-K)} \int dK'' \chi^{(2)}(K''|-K-K'') \left(\left\langle E^{(1)}_{K''} E^{(1)}_{-K-K''} E^{(1)}_{K'} E^{(1)}_{K-K'} \right\rangle - \left\langle E^{(1)}_{K'} E^{(1)}_{K-K'} \right\rangle \left\langle E^{(1)}_{K'} E^{(1)}_{K-K'} \right\rangle \right)$$

$$-\frac{1}{\varepsilon(K')} \int dK'' \chi^{(2)}(K''|K'-K'') \left(\left\langle E^{(1)}_{K''} E^{(1)}_{K'-K''} E^{(1)}_{-K} E^{(1)}_{K-K'} \right\rangle - \left\langle E^{(1)}_{K'} E^{(1)}_{K-K'} \right\rangle \left\langle E^{(1)}_{-K} E^{(1)}_{K-K'} \right\rangle \right)$$

$$-\frac{1}{\varepsilon(K-K')} \int dK'' \chi^{(2)}(K''|K-K'-K'') \left(\left\langle E^{(1)}_{K''} E^{(1)}_{K-K'-K''} E^{(1)}_{-K} E^{(1)}_{K'} \right\rangle - \left\langle E^{(1)}_{K'} E^{(1)}_{K-K'} \right\rangle \left\langle E^{(1)}_{-K} E^{(1)}_{K'} \right\rangle \right)$$

$$\langle E_{K_1} E_{K_2} E_{K_3} E_{K_4} \rangle = \delta(K_1 + K_2 + K_3 + K_4) \left[\delta(K_1 + K_2) \langle E_{K_1} E_{K_2} \rangle \langle E_{K_3} E_{K_4} \rangle \right.$$

$$\left. + \delta(K_1 + K_3) \langle E_{K_1} E_{K_3} \rangle \langle E_{K_2} E_{K_4} \rangle + \delta(K_1 + K_4) \langle E_{K_1} E_{K_4} \rangle \langle E_{K_2} E_{K_3} \rangle \right]$$

Particle Kinetic Equation

$$\frac{\partial f_e}{\partial t} = \frac{\partial}{\partial v_i}\left(A_i f_e + D_{ij}\frac{\partial f_e}{\partial v_j}\right),$$

$$A_i = \frac{e^2}{4\pi m_e}\int d\mathbf{k}\,\frac{k_i}{k^2}\sum_{\sigma=\pm1}\sigma\omega_{\mathbf{k}}^L\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v}),$$ ← Spontaneous drag (discrete particle effect)

$$D_{ij} = \frac{\pi e^2}{m_e^2}\int d\mathbf{k}\,\frac{k_i k_j}{k^2}\sum_{\sigma=\pm1}\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v})I_{\mathbf{k}}^{\sigma L},$$ ← Velocity space diffusion

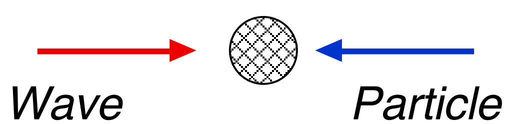
Wave Kinetic Equation

$$\frac{\partial I_{\mathbf{k}}^{\sigma L}}{\partial t} = \frac{\pi \omega_{pe}^2}{k^2} \int d\mathbf{v}\, \delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v}) \left(\frac{ne^2}{\pi} f_e + \sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}}^{\sigma L} \mathbf{k}\cdot \frac{\partial f_e}{\partial \mathbf{v}} \right)$$

Spontaneous emission

Induced emission
(Landau damping/
Quasi-linear growth/damping rate)

Linear wave-particle
resonance



$$\frac{\partial I_{\mathbf{k}}^{\sigma L}}{\partial t} = \frac{\pi \omega_{pe}^{2}}{k^{2}} \int d\mathbf{v}\, \delta(\sigma\omega_{\mathbf{k}}^{L} - \mathbf{k}\cdot\mathbf{v}) \left(\frac{ne^{2}}{\pi} f_{e} + \sigma\omega_{\mathbf{k}}^{L} I_{\mathbf{k}}^{\sigma L} \mathbf{k}\cdot \frac{\partial f_{e}}{\partial \mathbf{v}} \right)$$

$$+2 \sum_{\sigma',\sigma''=\pm 1} \sigma\omega_{\mathbf{k}}^{L} \int d\mathbf{k}' V_{\mathbf{k},\mathbf{k}'}^{L} \delta(\sigma\omega_{\mathbf{k}}^{L} - \sigma'\omega_{\mathbf{k}'}^{L} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{S})$$

$$\times \left(\sigma\omega_{\mathbf{k}}^{L} I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} - \sigma'\omega_{\mathbf{k}'}^{L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} I_{\mathbf{k}}^{\sigma L} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L} I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma L} \right)$$

Spontaneous decay

Induced decay

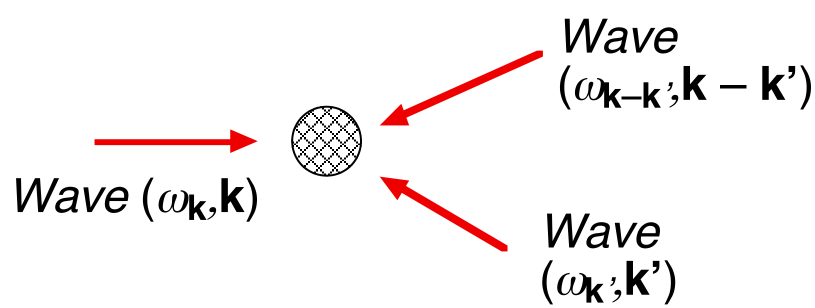
$$V_{\mathbf{k},\mathbf{k}'}^{L} = \frac{\pi}{4} \frac{e^{2}}{T_{e}^{2}} \frac{\mu_{\mathbf{k}-\mathbf{k}'} (\mathbf{k}\cdot\mathbf{k}')^{2}}{k^{2} k'^{2} |\mathbf{k}-\mathbf{k}'|^{2}}, \qquad \mu_{\mathbf{k}} = k^{3} \lambda_{De}^{3} \sqrt{\frac{m_{e}}{m_{i}}} \left(1 + \frac{3T_{i}}{T_{e}} \right)^{1/2}$$

$$\frac{\partial I_{\mathbf{k}}^{\sigma L}}{\partial t} = \frac{\pi \omega_{pe}^2}{k^2} \int d\mathbf{v}\, \delta(\sigma \omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v}) \left(\frac{ne^2}{\pi} f_e + \sigma \omega_{\mathbf{k}}^L I_{\mathbf{k}}^{\sigma L} \mathbf{k}\cdot \frac{\partial f_e}{\partial \mathbf{v}} \right)$$

$$+2 \sum_{\sigma',\sigma''=\pm 1} \sigma \omega_{\mathbf{k}}^L \int d\mathbf{k}' V_{\mathbf{k},\mathbf{k}'}^L \delta(\sigma \omega_{\mathbf{k}}^L - \sigma' \omega_{\mathbf{k}'}^L - \sigma'' \omega_{\mathbf{k}-\mathbf{k}'}^S)$$

$$\times \left(\sigma \omega_{\mathbf{k}}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} - \sigma' \omega_{\mathbf{k}'}^L I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} I_{\mathbf{k}}^{\sigma L} - \sigma'' \omega_{\mathbf{k}-\mathbf{k}'}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma L} \right)$$

Nonlinear wave-wave resonance



$$\frac{\partial I_{\mathbf{k}}^{\sigma L}}{\partial t} = \frac{\pi\omega_{pe}^2}{k^2}\int d\mathbf{v}\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v})\left(\frac{ne^2}{\pi}f_e + \sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}}^{\sigma L}\mathbf{k}\cdot\frac{\partial f_e}{\partial \mathbf{v}}\right)$$

$$+2\sum_{\sigma',\sigma''=\pm1}\sigma\omega_{\mathbf{k}}^L\int d\mathbf{k}' V_{\mathbf{k},\mathbf{k}'}^L\delta(\sigma\omega_{\mathbf{k}}^L - \sigma'\omega_{\mathbf{k}'}^L - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^S)$$

$$\times\left(\sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} - \sigma'\omega_{\mathbf{k}'}^L I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} I_{\mathbf{k}}^{\sigma L} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma L}\right)$$

$$-\frac{\pi e^2}{m_e^2\omega_{pe}^2}\sigma\omega_{\mathbf{k}}^L\sum_{\sigma'=\pm1}\int d\mathbf{k}'\int d\mathbf{v}\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2 k'^2}\delta[\sigma\omega_{\mathbf{k}}^L - \sigma'\omega_{\mathbf{k}'}^L - (\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}]$$

$$\times\left(\frac{ne^2}{\pi\omega_{pe}^2}(\sigma'\omega_{\mathbf{k}'}^L I_{\mathbf{k}}^{\sigma L} - \sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}'}^{\sigma' L})(f_e + f_i)\right.$$

← Spontaneous scattering

$$\left.+(\sigma'\omega_{\mathbf{k}'}^L - \sigma\omega_{\mathbf{k}}^L)I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma L}(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial f_e}{\partial \mathbf{v}} - \frac{m_e}{m_i}I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma L}(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial f_i}{\partial \mathbf{v}}\right)$$

Induced scattering (NL Landau damping)

↑ (scattering off thermal ions)

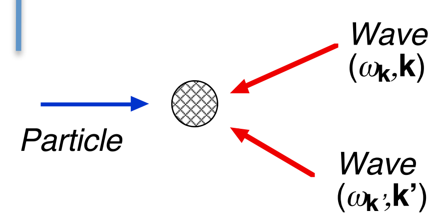
$$\frac{\partial I_{\mathbf{k}}^{\sigma L}}{\partial t} = \frac{\pi \omega_{pe}^2}{k^2} \int d\mathbf{v}\, \delta(\sigma \omega_{\mathbf{k}}^L - \mathbf{k}\cdot \mathbf{v}) \left(\frac{ne^2}{\pi} f_e + \sigma \omega_{\mathbf{k}}^L I_{\mathbf{k}}^{\sigma L} \mathbf{k}\cdot \frac{\partial f_e}{\partial \mathbf{v}} \right)$$

$$+2 \sum_{\sigma',\sigma''=\pm 1} \sigma \omega_{\mathbf{k}}^L \int d\mathbf{k}' V_{\mathbf{k},\mathbf{k}'}^L \delta(\sigma \omega_{\mathbf{k}}^L - \sigma' \omega_{\mathbf{k}'}^L - \sigma'' \omega_{\mathbf{k}-\mathbf{k}'}^S)$$

$$\times \left(\sigma \omega_{\mathbf{k}}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} - \sigma' \omega_{\mathbf{k}'}^L I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} I_{\mathbf{k}}^{\sigma L} - \sigma'' \omega_{\mathbf{k}-\mathbf{k}'}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma L} \right)$$

$$-\frac{\pi e^2}{m_e^2 \omega_{pe}^2} \sigma \omega_{\mathbf{k}}^L \sum_{\sigma'=\pm 1} \int d\mathbf{k}' \int d\mathbf{v} \frac{(\mathbf{k}\cdot \mathbf{k}')^2}{k^2 k'^2} \boxed{\delta[\sigma \omega_{\mathbf{k}}^L - \sigma' \omega_{\mathbf{k}'}^L - (\mathbf{k}-\mathbf{k}')\cdot \mathbf{v}]}$$

$$\times \left(\frac{ne^2}{\pi \omega_{pe}^2} (\sigma' \omega_{\mathbf{k}'}^L I_{\mathbf{k}}^{\sigma L} - \sigma \omega_{\mathbf{k}}^L I_{\mathbf{k}'}^{\sigma' L})(f_e + f_i) \right.$$



$$\left. +(\sigma' \omega_{\mathbf{k}'}^L - \sigma \omega_{\mathbf{k}}^L) I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma L} (\mathbf{k}-\mathbf{k}')\cdot \frac{\partial f_e}{\partial \mathbf{v}} - \frac{m_e}{m_i} I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma L} (\mathbf{k}-\mathbf{k}')\cdot \frac{\partial f_i}{\partial \mathbf{v}} \right)$$

Nonlinear wave-particle resonance

Weak turbulence theory

- L. M. Gorbunov, V. V. Pustovalov, and V. P. Silin, *Sov. Phys. JETP* **20**, 967 (1965)
- L. M. Al'tshul' and V. I. Karpman, *Sov Phys. JETP* **20**, 1043 (1965)
- L. M. Kovrizhnykh, *Sov. Phys. JETP* **21**, 744 (1965)
- B. B. Kadomtsev, *Plasma Turbulence* (Academic Press, 1965)
- V. N. Tsytovich, *Sov. Phys. USPEKHI* **9**, 805 (1967)
- V. N. Tsytovich, *Nonlinear Effects in Plasma* (Plenum Press, 1970)
- R. C. Davidson, *Methods in Nonlinear Plasma Theory* (Academic Press, 1972)
- V. N. Tsytovich, *Theory of Turbulent Plasma* (Consultants Bureau, 1977)
- D. B. Melrose, *Plasma Astrophysics* (Gordon and Breach, 1980)
- A. G. Sitenko, *Fluctuations and Non-Linear Wave Interactions in Plasmas* (Pergamon, 1982)

Eq. of weak turbulence theory

$$\frac{\partial f_e}{\partial t} = \frac{\partial}{\partial v_i}\left(A_i f_e + D_{ij}\frac{\partial f_e}{\partial v_j}\right),$$

$$A_i = \frac{e^2}{4\pi m_e}\int d\mathbf{k}\,\frac{k_i}{k^2}\sum_{\sigma=\pm1}\sigma\omega_{\mathbf{k}}^L\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v}),$$

$$D_{ij} = \frac{\pi e^2}{m_e^2}\int d\mathbf{k}\,\frac{k_i k_j}{k^2}\sum_{\sigma=\pm1}\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v})I_{\mathbf{k}}^{\sigma L}.$$

$$\frac{\partial I_{\mathbf{k}}^{\sigma L}}{\partial t} = \frac{\pi\omega_{pe}^2}{k^2}\int d\mathbf{v}\,\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v})\left(\frac{ne^2}{\pi}f_e + \sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}}^{\sigma L}\mathbf{k}\cdot\frac{\partial f_e}{\partial \mathbf{v}}\right)$$

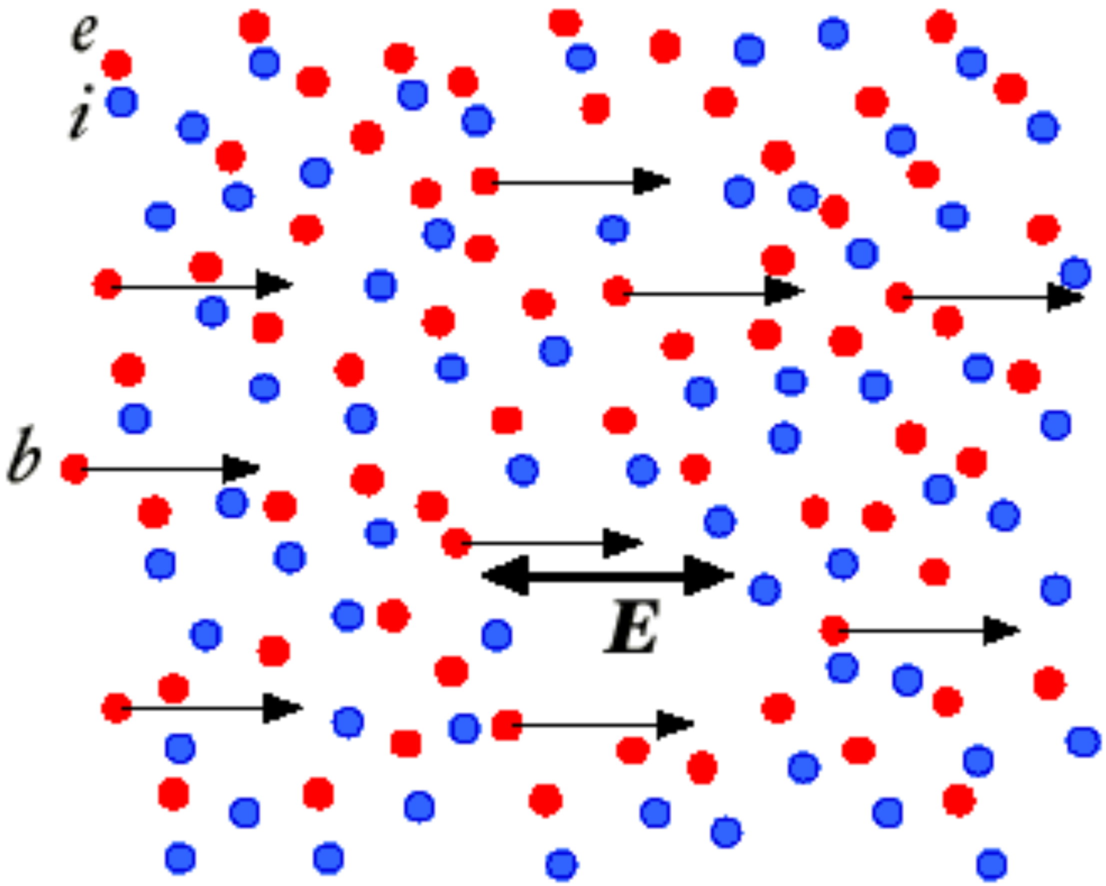
$$+2\sum_{\sigma',\sigma''=\pm1}\sigma\omega_{\mathbf{k}}^L\int d\mathbf{k}'V_{\mathbf{k},\mathbf{k}'}^L\delta(\sigma\omega_{\mathbf{k}}^L - \sigma'\omega_{\mathbf{k}'}^L - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^S)$$

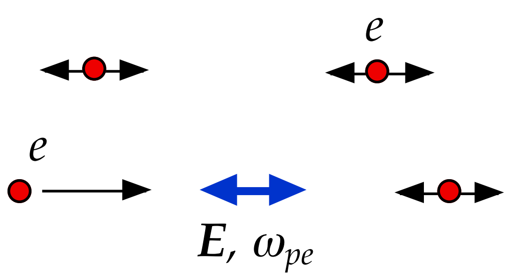
$$\times\left(\sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} - \sigma'\omega_{\mathbf{k}'}^L I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} I_{\mathbf{k}}^{\sigma L} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma L}\right)$$

$$-\frac{\pi e^2}{m_e^2\omega_{pe}^2}\sigma\omega_{\mathbf{k}}^L\sum_{\sigma'=\pm1}\int d\mathbf{k}'\int d\mathbf{v}\,\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2 k'^2}\delta[\sigma\omega_{\mathbf{k}}^L - \sigma'\omega_{\mathbf{k}'}^L - (\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}]$$

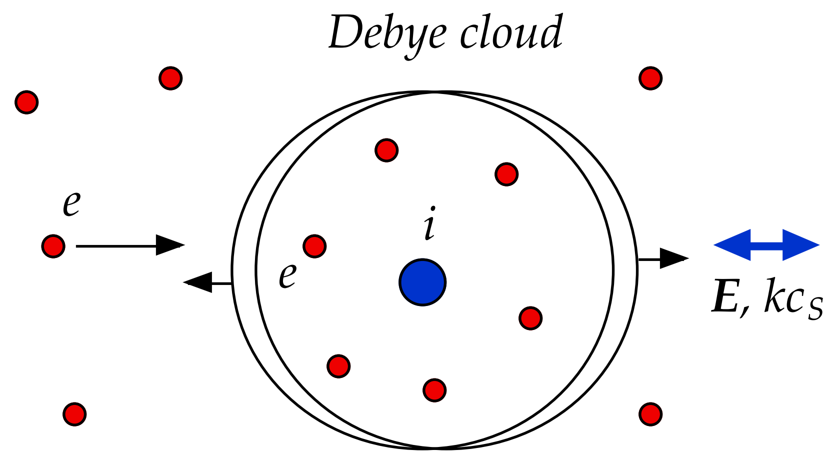
$$\times\left(\frac{ne^2}{\pi\omega_{pe}^2}(\sigma'\omega_{\mathbf{k}'}^L I_{\mathbf{k}}^{\sigma L} - \sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}'}^{\sigma' L})f_i - \frac{m_e}{m_i}I_{\mathbf{k}'}^{\sigma' L}I_{\mathbf{k}}^{\sigma L}(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial f_i}{\partial \mathbf{v}}\right)$$

Beam-generated Langmuir turbulence





Langmuir oscillation

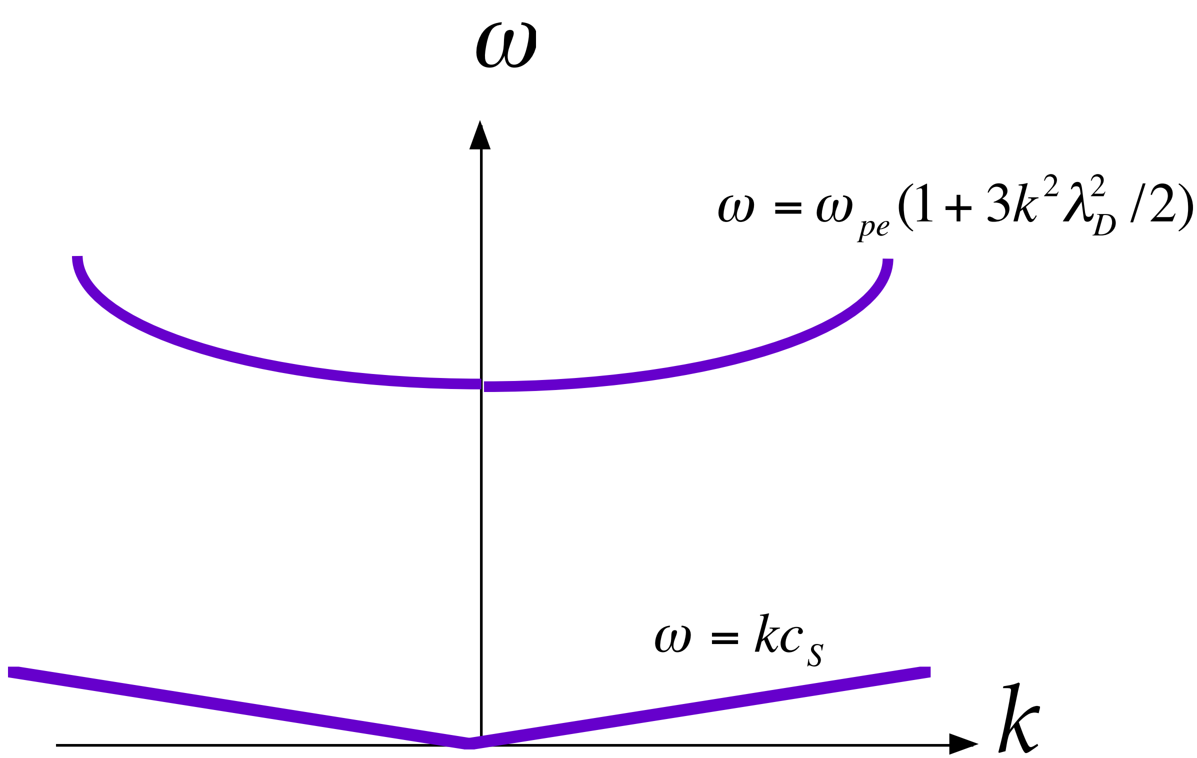


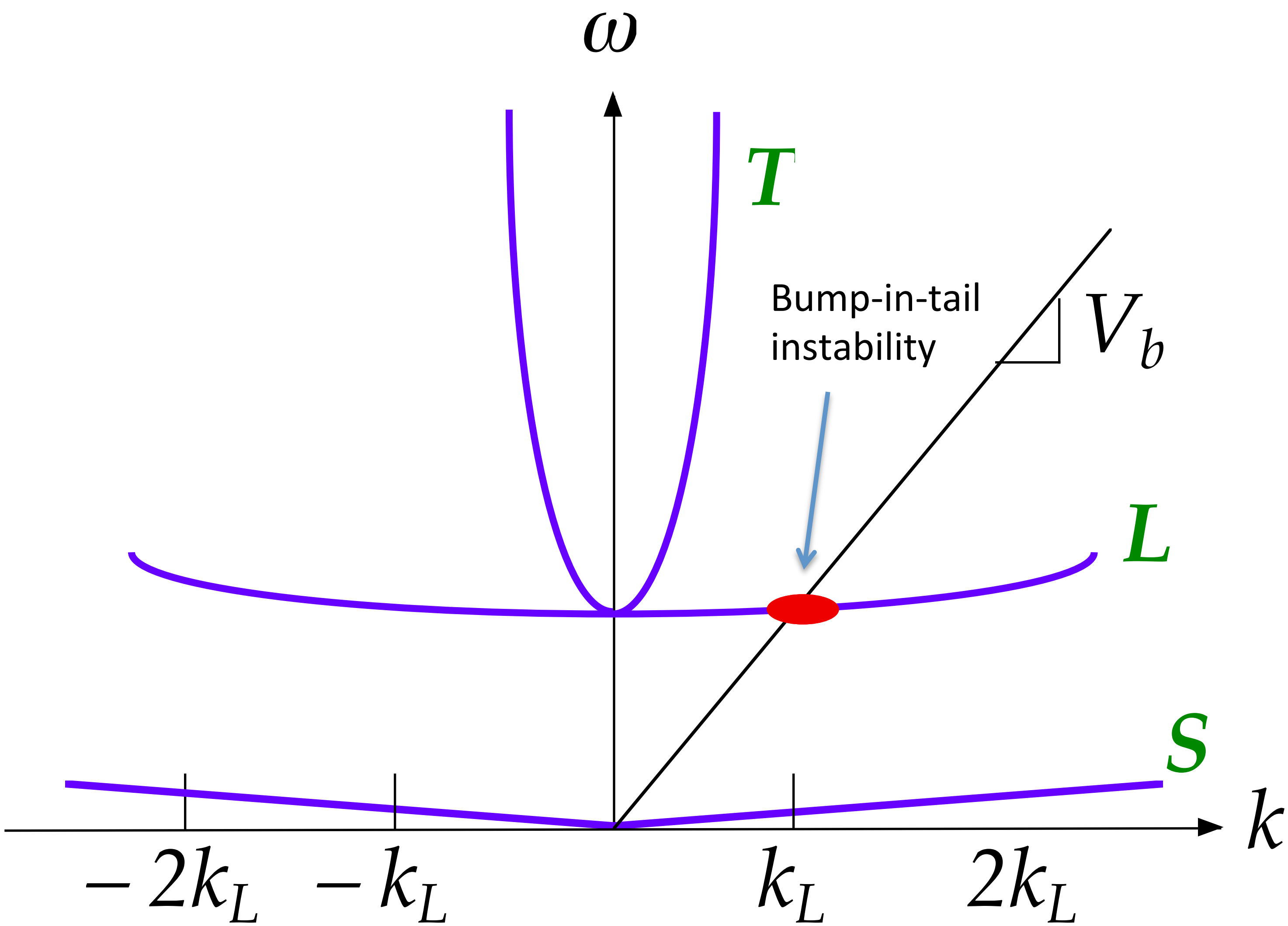
Ion-sound wave

$$\mathbf{E}(\mathbf{x},t) = \mathbf{E}\cos(\mathbf{k} \bullet \mathbf{x} - \omega t),$$

$$\omega = \omega_{pe}\left(1 + \frac{3}{2}k^2\lambda_D^2\right) = \sqrt{\frac{4\pi ne^2}{m_e}}\left(1 + k^2\,\frac{3T_e}{4\pi ne^2}\right), \quad or$$

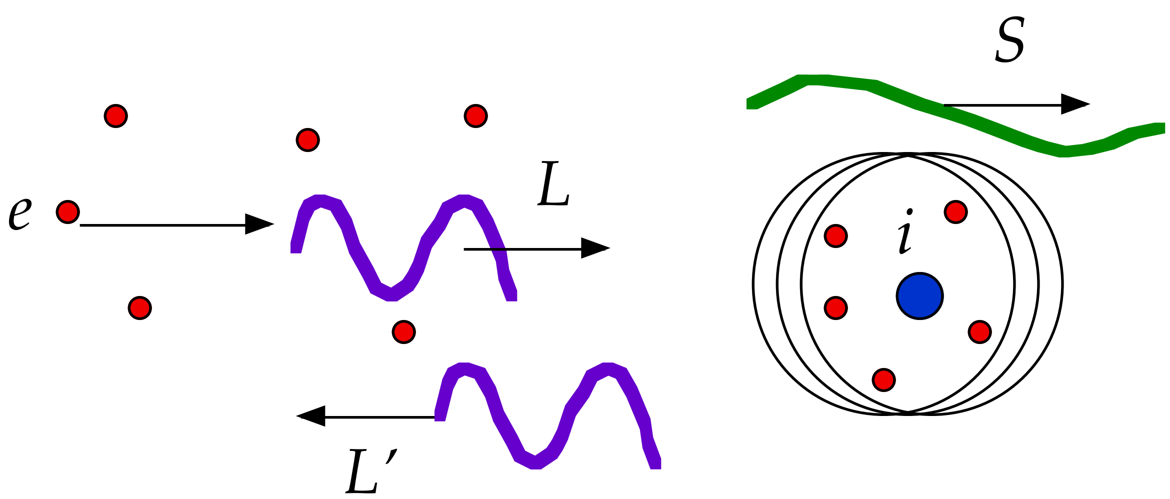
$$\omega = kc_S = k\sqrt{\frac{T_e}{m_i}}.$$

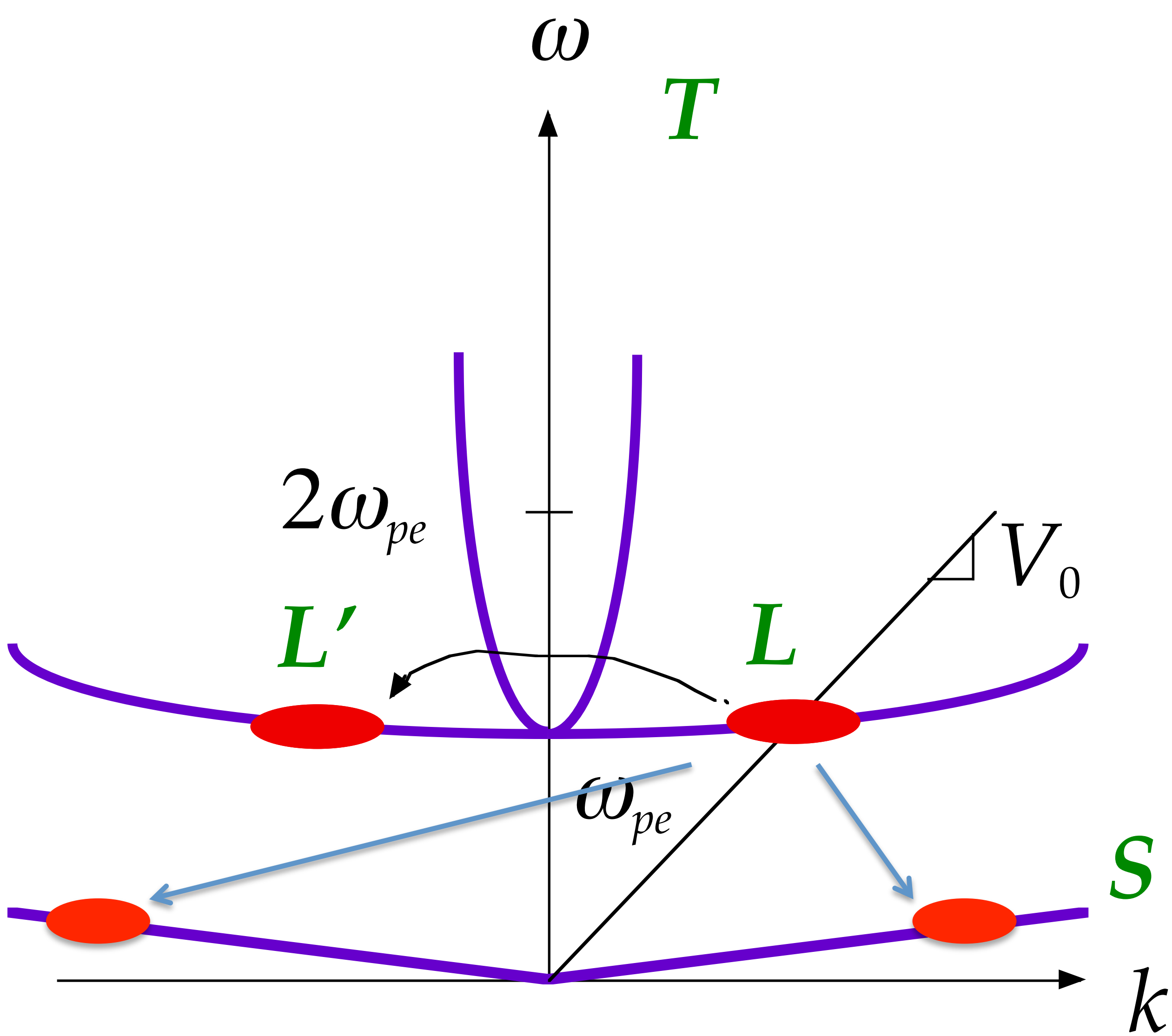




ω
T
Bump-in-tail
instability
V_b
L
S
k
$-2k_L$
$-k_L$
k_L
$2k_L$

Backscattered L wave





ω
T
2ω_pe
V_0
L′
L
ω_pe
S
k

Eq. of weak turbulence theory

$$\frac{\partial f_e}{\partial t} = \frac{\partial}{\partial v_i}\left(A_i f_e + D_{ij}\frac{\partial f_e}{\partial v_j}\right),$$

$$A_i = \frac{e^2}{4\pi m_e}\int d\mathbf{k}\,\frac{k_i}{k^2}\sum_{\sigma=\pm1}\sigma\omega_{\mathbf{k}}^L\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v}),$$

$$D_{ij} = \frac{\pi e^2}{m_e^2}\int d\mathbf{k}\,\frac{k_i k_j}{k^2}\sum_{\sigma=\pm1}\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v})I_{\mathbf{k}}^{\sigma L}.$$

$$\frac{\partial I_{\mathbf{k}}^{\sigma L}}{\partial t} = \frac{\pi\omega_{pe}^2}{k^2}\int d\mathbf{v}\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v})\left(\frac{ne^2}{\pi}f_e + \sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}}^{\sigma L}\mathbf{k}\cdot\frac{\partial f_e}{\partial\mathbf{v}}\right)$$

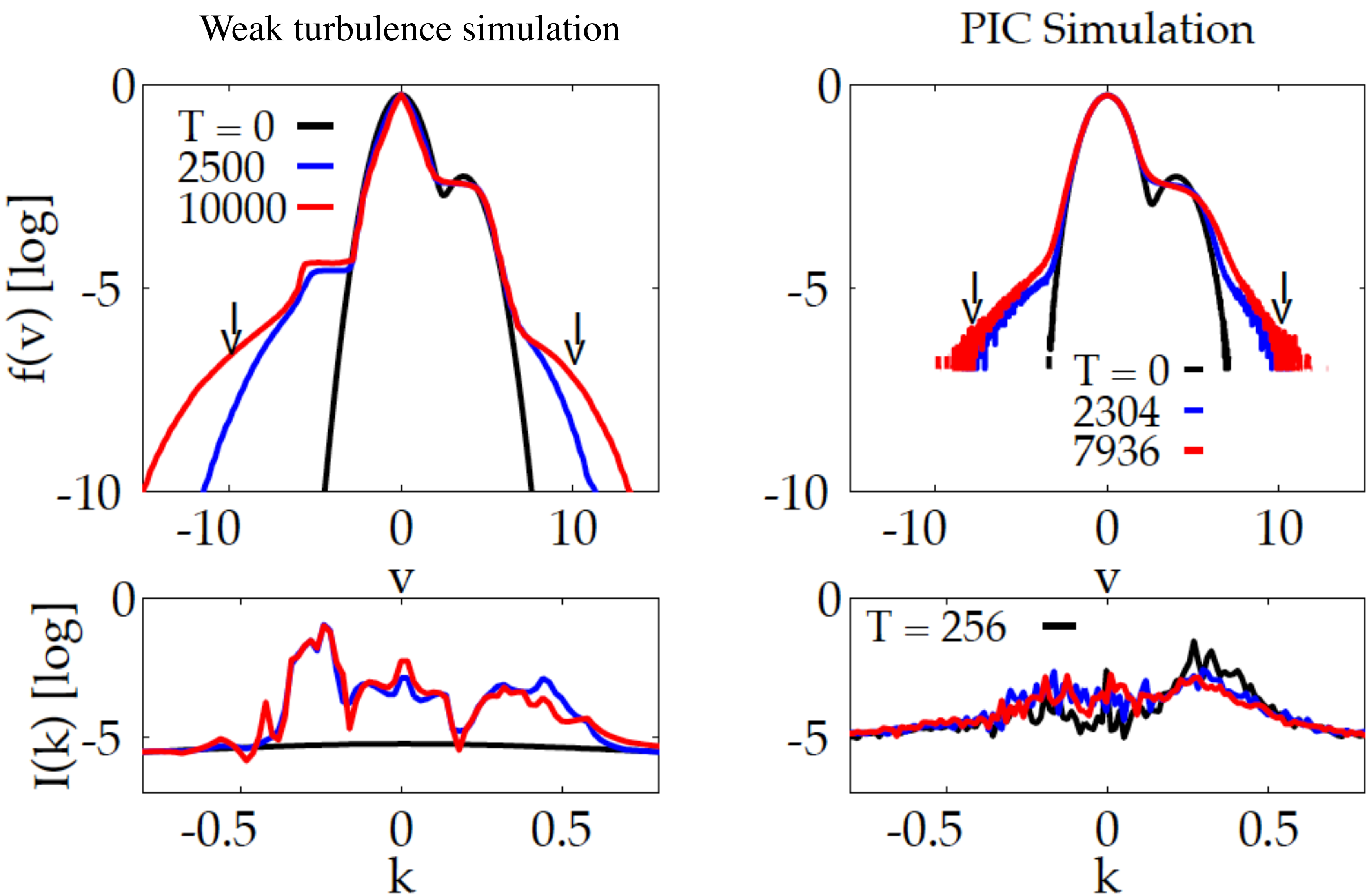
$$+2\sum_{\sigma',\sigma''=\pm1}\sigma\omega_{\mathbf{k}}^L\int d\mathbf{k}' V_{\mathbf{k},\mathbf{k}'}^L\delta(\sigma\omega_{\mathbf{k}}^L - \sigma'\omega_{\mathbf{k}'}^L - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^S)$$

$$\times\left(\sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} - \sigma'\omega_{\mathbf{k}'}^L I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} I_{\mathbf{k}}^{\sigma L} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma L}\right)$$

$$-\frac{\pi e^2}{m_e^2\omega_{pe}^2}\sigma\omega_{\mathbf{k}}^L\sum_{\sigma'=\pm1}\int d\mathbf{k}'\int d\mathbf{v}\,\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2 k'^2}\delta[\sigma\omega_{\mathbf{k}}^L - \sigma'\omega_{\mathbf{k}'}^L - (\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}]$$

$$\times\left(\frac{ne^2}{\pi\omega_{pe}^2}(\sigma'\omega_{\mathbf{k}'}^L I_{\mathbf{k}}^{\sigma L} - \sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}'}^{\sigma' L})f_i - \frac{m_e}{m_i}I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma L}(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial f_i}{\partial\mathbf{v}}\right)$$

Beam-Plasma Instability [Ryu et al., 2007]

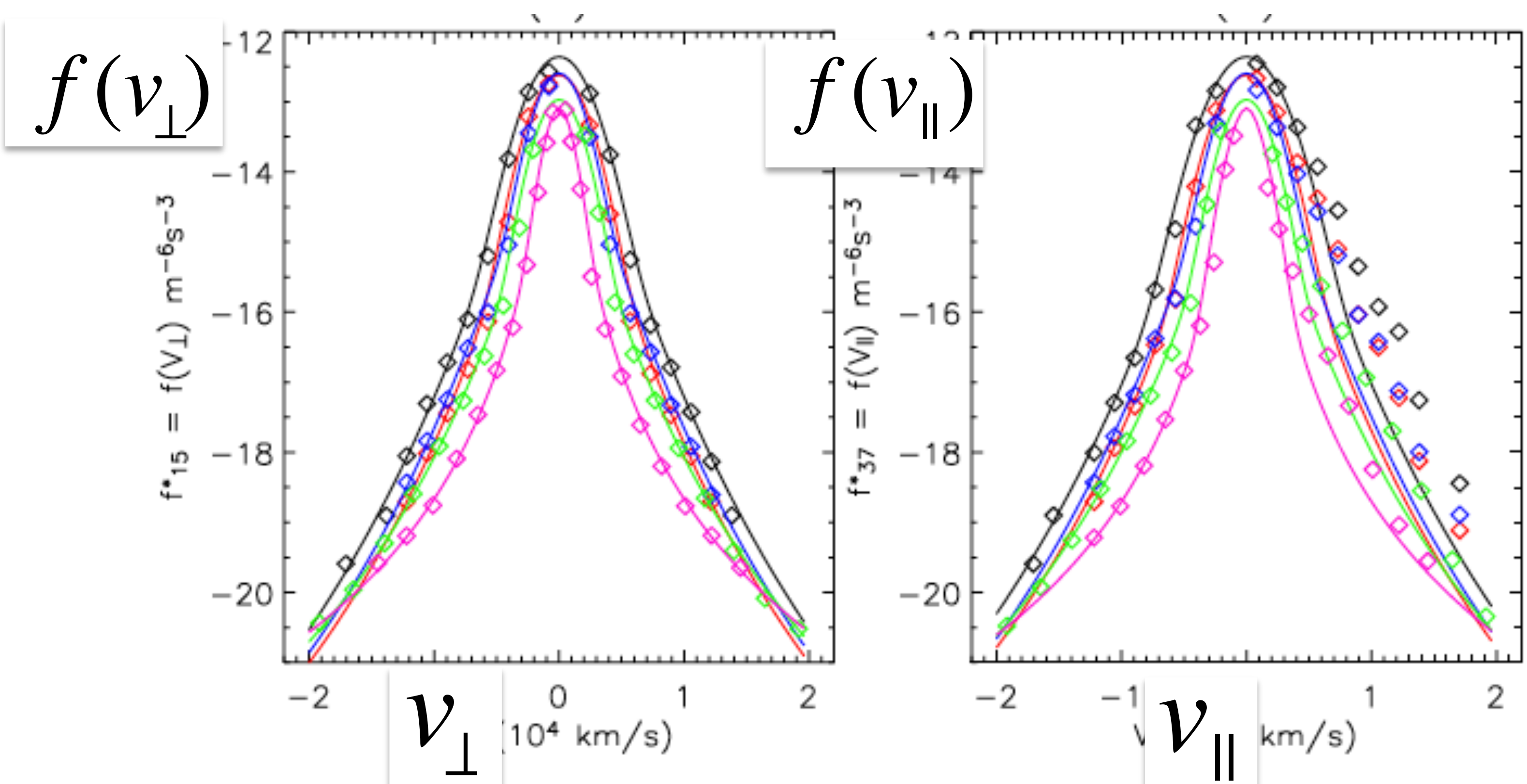


Maksimovic et al. *JGR*, *110*, A09104 (2005)

$$f = n_c f_c + n_h f_h,$$

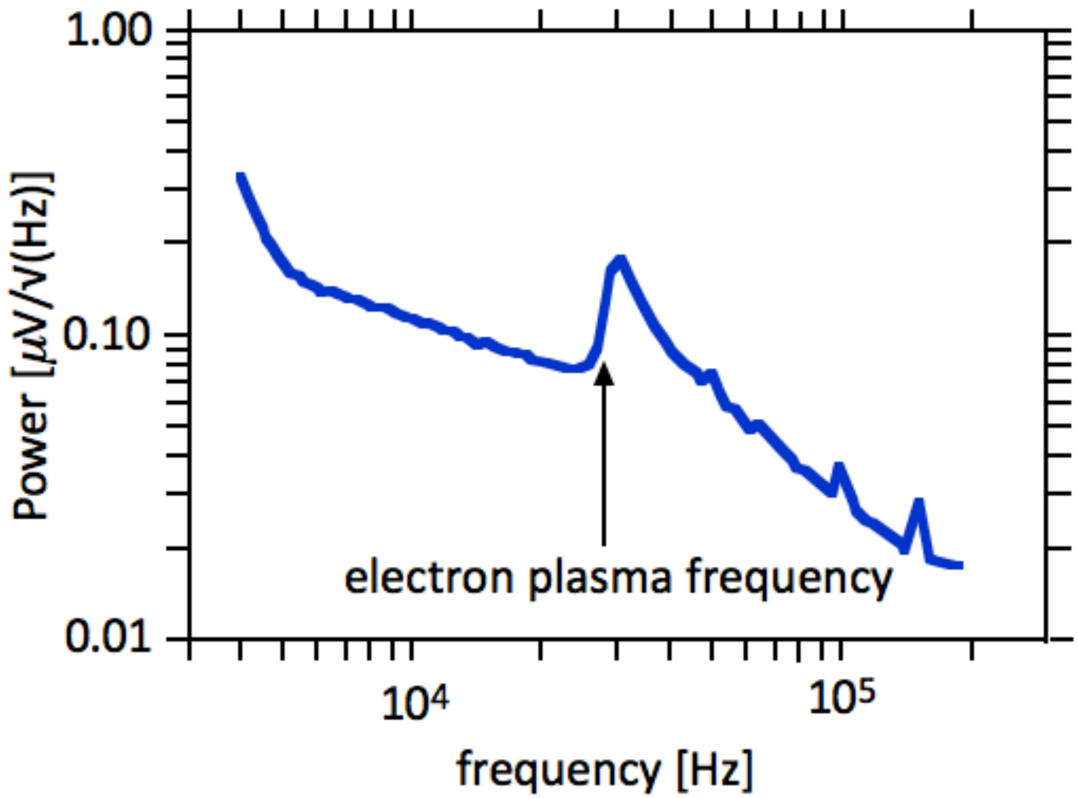
(bi) kappa model

$$f_c \propto \exp\left(-\frac{v_\perp^2}{\alpha_{\perp c}^2} - \frac{v_\parallel^2}{\alpha_{\parallel c}^2}\right), \qquad f_h \propto \frac{1}{\left(1 + \frac{v_\perp^2}{\kappa\alpha_{\perp h}^2} + \frac{v_\parallel^2}{\kappa\alpha_{\parallel h}^2}\right)^{-\kappa}}$$



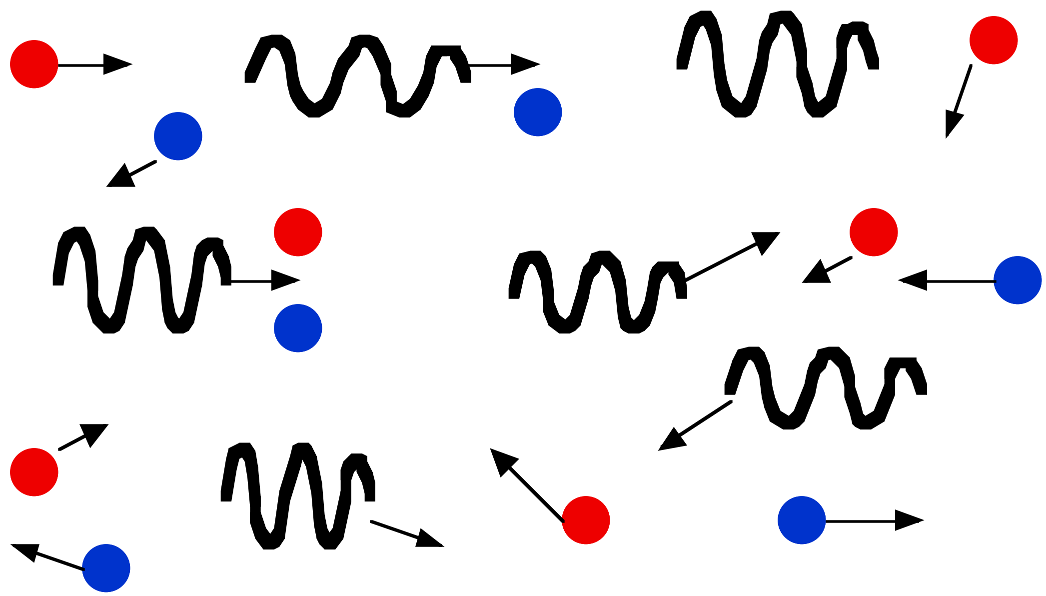
$I(k)$

Langmuir
Turbulence
Spectrum



Quasi-thermal noise [by Stuart Bale]

Turbulent Equilibrium



Particles and wave constantly exchange momentum and energy but are in dynamical steady state.

Asymptotic solution ($\partial/\partial t = 0$)

$$\frac{\partial f_e}{\partial t} = \frac{\partial}{\partial v_i}\left(A_i f_e + D_{ij}\frac{\partial f_e}{\partial v_j}\right),$$

$$A_i = \frac{e^2}{4\pi m_e}\int d\mathbf{k}\,\frac{k_i}{k^2}\sum_{\sigma=\pm1}\sigma\omega_{\mathbf{k}}^L\,\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v}),$$

$$D_{ij} = \frac{\pi e^2}{m_e^2}\int d\mathbf{k}\,\frac{k_i k_j}{k^2}\sum_{\sigma=\pm1}\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v})I_{\mathbf{k}}^{\sigma L}.$$

Linear wave-particle

$$\frac{\partial I_{\mathbf{k}}^{\sigma L}}{\partial t} = \frac{\pi\omega_{pe}^2}{k^2}\int d\mathbf{v}\,\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v})\left(\frac{ne^2}{\pi}f_e + \sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}}^{\sigma L}\mathbf{k}\cdot\frac{\partial f_e}{\partial \mathbf{v}}\right)$$

Wave-wave

$$+2\sum_{\sigma',\sigma''=\pm1}\sigma\omega_{\mathbf{k}}^L\int d\mathbf{k}' V_{\mathbf{k},\mathbf{k}'}^L\,\delta(\sigma\omega_{\mathbf{k}}^L - \sigma'\omega_{\mathbf{k}'}^L - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^S)$$
$$\times\left(\sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} - \sigma'\omega_{\mathbf{k}'}^L I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} I_{\mathbf{k}}^{\sigma L} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma L}\right)$$

Nonlinear wave-particle

$$-\frac{\pi e^2}{m_e^2\omega_{pe}^2}\sigma\omega_{\mathbf{k}}^L\sum_{\sigma'=\pm1}\int d\mathbf{k}'\int d\mathbf{v}\,\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2 k'^2}\delta[\sigma\omega_{\mathbf{k}}^L - \sigma'\omega_{\mathbf{k}'}^L - (\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}]$$
$$\times\left(\frac{ne^2}{\pi\omega_{pe}^2}(\sigma'\omega_{\mathbf{k}'}^L I_{\mathbf{k}}^{\sigma L} - \sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}'}^{\sigma' L})f_i - \frac{m_e}{m_i}I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma L}(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial f_i}{\partial \mathbf{v}}\right)$$

Self-consistent, steady-state electron velocity and Langmuir spectrum distributions

$$\frac{\partial f}{\partial t} = 0 = \frac{\pi e^2}{m^2} \int \frac{d\mathbf{k}}{k^2} \mathbf{k} \bullet \frac{\partial}{\partial \mathbf{v}} \delta(\omega_{\mathbf{k}}^L - \mathbf{k} \bullet \mathbf{v})$$

$$\times \left(\frac{m}{4\pi^2} \omega_{\mathbf{k}}^L f + I(\mathbf{k}) \mathbf{k} \bullet \frac{\partial f}{\partial \mathbf{v}} \right),$$

$$\frac{\partial I(\mathbf{k})}{\partial t} = 0 = \frac{\pi \omega_p^2}{k^2} \int d\mathbf{v} \delta(\omega_{\mathbf{k}}^L - \mathbf{k} \bullet \mathbf{v})$$

$$\times \left(\frac{n e^2}{\pi} f + \omega_{\mathbf{k}}^L I(\mathbf{k}) \mathbf{k} \bullet \frac{\partial f}{\partial \mathbf{v}} \right).$$

Electron VDF (cont'd) [cf, Hasegawa et al., 1985]

$$f(v) = C \exp\left(-\frac{m}{4\pi^2} \int dv \frac{\int_{\omega_p/v}^{\infty} \frac{dk}{k}}{\int_{\omega_p/v}^{\infty} \frac{I(k)dk}{k}} \right).$$

e.g., thermal equilibrium

$$I(k) \propto T, \quad f(v) \propto \exp\left(-\frac{mv^2}{2T} \right)$$

Self-Consistent Solution: Turbulent Equilibrium

$$f = \frac{1}{\pi^{3/2}\alpha^3} \frac{\Gamma(\kappa+1)}{\kappa^{3/2}\Gamma(\kappa-1/2)} \frac{1}{\left(1+\frac{v^2}{\kappa\alpha^2}\right)^{\kappa+1}},$$

$$I(k) = \frac{T_e}{4\pi^2} \frac{\kappa-3/2}{\kappa+1} \left(1+\frac{1}{(\kappa-3/2)(k\alpha/\omega_p)^2}\right).$$

- Self-consistent steady-state particle and wave distributions leads to kappa distribution, but κ is undetermined:

$$\frac{\partial f_e}{\partial t} = \frac{\partial}{\partial v_i}\left(A_i f_e + D_{ij}\frac{\partial f_e}{\partial v_j}\right),$$

Asymptotic solution (∂/∂*t* = 0)

$$A_i = \frac{e^2}{4\pi m_e}\int d\mathbf{k}\,\frac{k_i}{k^2}\sum_{\sigma=\pm1}\sigma\omega_{\mathbf{k}}^L\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v}),$$

$$D_{ij} = \frac{\pi e^2}{m_e^2}\int d\mathbf{k}\,\frac{k_i k_j}{k^2}\sum_{\sigma=\pm1}\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v})I_{\mathbf{k}}^{\sigma L}.$$

Linear wave-particle

$$\frac{\partial I_{\mathbf{k}}^{\sigma L}}{\partial t} = \frac{\pi\omega_{pe}^2}{k^2}\int d\mathbf{v}\,\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v})\left(\frac{ne^2}{\pi}f_e + \sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}}^{\sigma L}\mathbf{k}\cdot\frac{\partial f_e}{\partial \mathbf{v}}\right)$$

Wave-wave

$$+2\sum_{\sigma',\sigma''=\pm1}\sigma\omega_{\mathbf{k}}^L\int d\mathbf{k}' V_{\mathbf{k},\mathbf{k}'}^L\delta(\sigma\omega_{\mathbf{k}}^L - \sigma'\omega_{\mathbf{k}'}^L - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^S)$$
$$\times\left(\sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} - \sigma'\omega_{\mathbf{k}'}^L I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} I_{\mathbf{k}}^{\sigma L} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma L}\right)$$

Nonlinear wave-particle

$$-\frac{\pi e^2}{m_e^2\omega_{pe}^2}\sigma\omega_{\mathbf{k}}^L\sum_{\sigma'=\pm1}\int d\mathbf{k}'\int d\mathbf{v}\,\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2}\delta[\sigma\omega_{\mathbf{k}}^L - \sigma'\omega_{\mathbf{k}'}^L - (\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}]$$
$$\times\left(\frac{ne^2}{\pi\omega_{pe}^2}(\sigma'\omega_{\mathbf{k}'}^L I_{\mathbf{k}}^{\sigma L} - \sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}'}^{\sigma' L})f_i - \frac{m_e}{m_i}I_{\mathbf{k}'}^{\sigma' L}I_{\mathbf{k}}^{\sigma L}(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial f_i}{\partial \mathbf{v}}\right)$$

Balance of nonlinear terms within wave kinetic equation

$$0 = \frac{\pi\omega_{pe}^2}{k^2}\int d\mathbf{v}\,\delta(\omega_k - \mathbf{k}\cdot\mathbf{v})\left(\frac{ne^2}{\pi}f_e + \omega_{pe}I(\mathbf{k})\mathbf{k}\cdot\frac{\partial f_e}{\partial\mathbf{v}}\right)$$

$$-\frac{\omega_{pe}}{4\pi nT_i}\int d\mathbf{k}'\int d\mathbf{v}\,\delta[\omega_k - \omega_{k'} - (\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}]$$

$$\times\left(\frac{T_i}{4\pi^2}[\omega_{k'}I(\mathbf{k}) - \omega_k I(\mathbf{k}')] + I(\mathbf{k})I(\mathbf{k}')(\omega_k - \omega_{k'})\right)f_i\,.$$

Balance of spontaneous and induced *emissions*

$$0 = \frac{\pi\omega_{pe}^2}{k^2}\int d\mathbf{v}\,\delta(\omega_k - \mathbf{k}\cdot\mathbf{v})\left(\frac{ne^2}{\pi}f_e + \omega_{pe}I(\mathbf{k})\mathbf{k}\cdot\frac{\partial f_e}{\partial \mathbf{v}}\right) \quad =0$$

$$-\frac{\omega_{pe}}{4\pi n T_i}\int d\mathbf{k}'\int d\mathbf{v}\,\delta[\omega_k - \omega_{k'} - (\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}]$$

$$\times\left(\frac{T_i}{4\pi^2}[\omega_{k'}I(\mathbf{k}) - \omega_k I(\mathbf{k}')] + I(\mathbf{k})I(\mathbf{k}')(\omega_k - \omega_{k'})\right)f_i. \quad =0$$

Balance of spontaneous and induced *scatterings*

$$\omega_k \frac{dI(k)}{dk} + \frac{4\pi^2}{T_i} \frac{d\omega_k}{dk} [I(k)]^2 - \frac{d\omega_k}{dk} I(k) = 0.$$

Solving the above equation one obtains the steady-state turbulence intensity that balances the spontaneous and induced scattering processes:

$$I(k) = \frac{T_i}{4\pi^2}\left(1 + \frac{4}{3}\frac{\omega_{pe}^2}{k^2 v_{Te}^2}\right).$$

• **We now have two alternative expressions for the steady-state turbulence intensity:**

$$I(k) = \frac{T_e}{4\pi^2}\frac{\kappa - 3/2}{\kappa + 1}\left(1 + \frac{\omega_{pe}^2}{(\kappa - 3/2)k^2 v_{Te}^2}\right),$$

$$I(k) = \frac{T_i}{4\pi^2}\left(1 + \frac{4}{3}\frac{\omega_{pe}^2}{k^2 v_{Te}^2}\right).$$

• Balance of spontaneous and induced emissions

• Balance of spontaneous and induced scatterings

• The only way the two expressions can be reconciled is when

$$T_e \frac{\kappa - 3/2}{\kappa + 1} = T_i, \quad and \quad \kappa - \frac{3}{2} = \frac{3}{4}$$

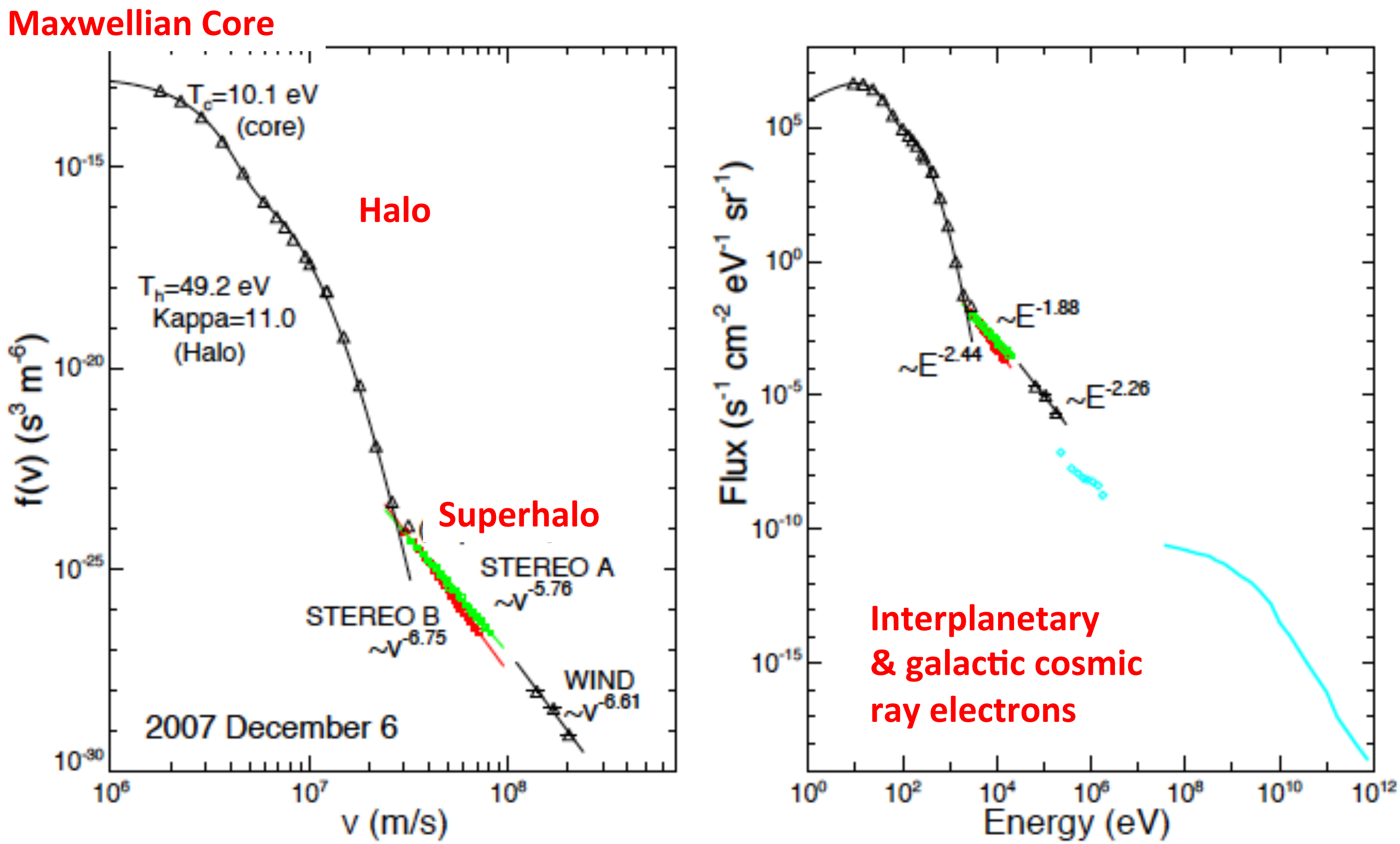
- **Turbulent equilibrium:**

$$f_e(v) = \frac{1}{\pi^{3/2} v_{Te}^3} \frac{\Gamma(\kappa+1)}{\kappa^{3/2}\Gamma(\kappa-1/2)} \frac{1}{[1+v^2/\kappa v_{Te}^2]^{\kappa+1}},$$

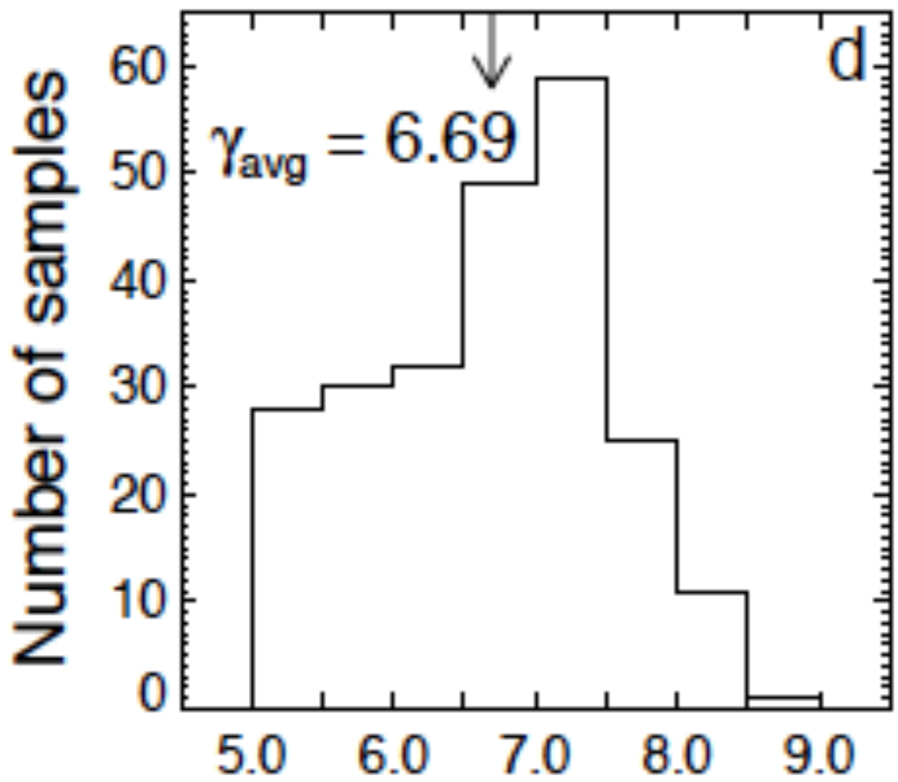
$$I(k) = \frac{T_e}{4\pi^2} \frac{\kappa-3/2}{\kappa+1}\left(1+\frac{\omega_{pe}^2}{(\kappa-3/2)(kv_{Te})^2}\right).$$

$$\kappa = \frac{9}{4} = 2.25.$$

SOLAR WIND ELECTRON DISTRIBUTION AT 1 AU



Wang et al., *ApJ Lett.* (2012)



$$f_{observation} \propto \nu^{-\gamma}, \qquad f_{theory} \propto \nu^{-2\kappa-2} \sim \nu^{-6.5},$$

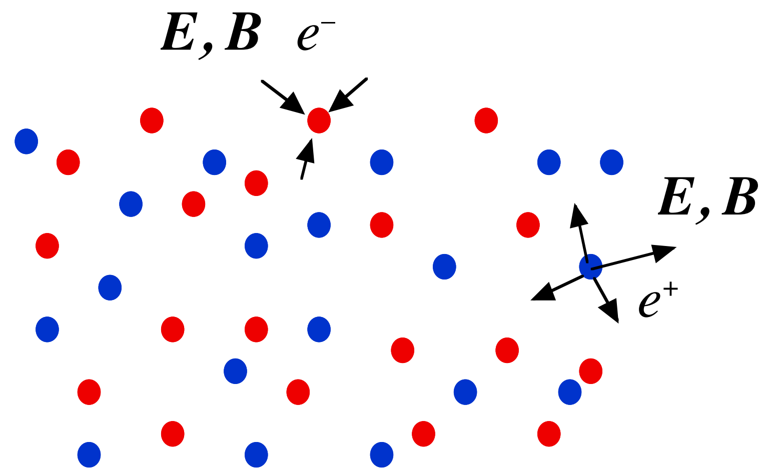
$$\gamma_{avg}^{observation} \cong 6.69 \qquad \gamma_{theory} \cong 6.5$$

Part 1. Conclusion

- Perturbative nonlinear kinetic theory of Langmuir turbulence is well known [e.g., Tytovich, 1970; Davidson 1972; Melrose, 1980; Sitenko, 1982, etc.]

- Application of the theory to explain quiet time solar wind electron kappa distribution [Yoon, 2014] is, however, new. Comparison between theory (κ = 6.5) and observation (κ_{avg} = 6.69) is favorable.

Part II. Electromagnetic Problem



$$N_a(\mathbf{r},\mathbf{v},t) = \sum_{i=1}^{N} \delta[\mathbf{r} - \mathbf{r}_i^a(t)]\delta[\mathbf{v} - \mathbf{v}_i^a(t)],$$

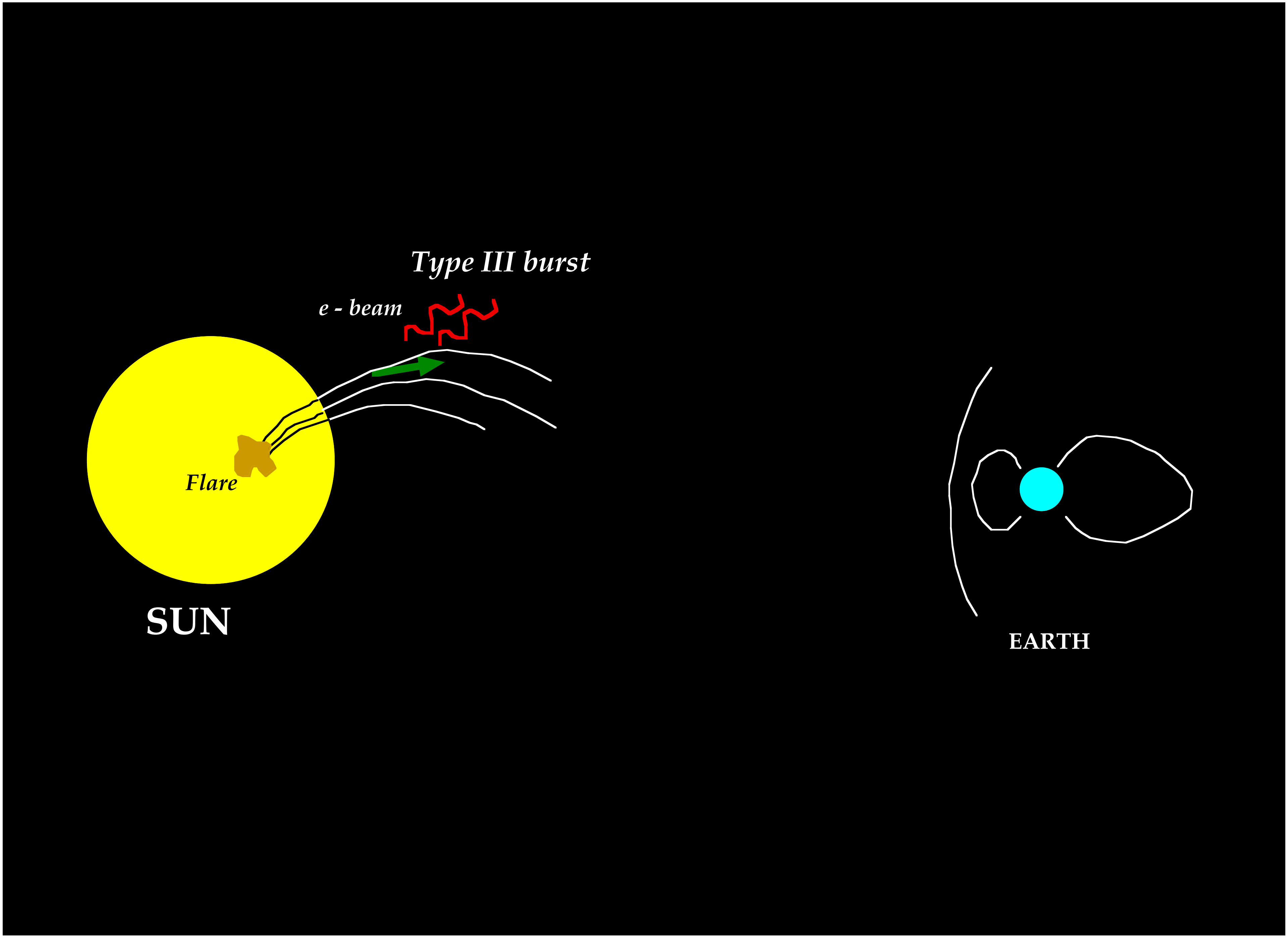
$$\left[\frac{\partial}{\partial t} + \mathbf{v} \bullet \nabla + \frac{e_a}{m_a}\left(\mathbf{E}(\mathbf{r},t) + \frac{\mathbf{v}}{c} \times \mathbf{B}(\mathbf{r},t)\right) \bullet \frac{\partial}{\partial \mathbf{v}}\right] N_a(\mathbf{r},\mathbf{v},t) = 0,$$

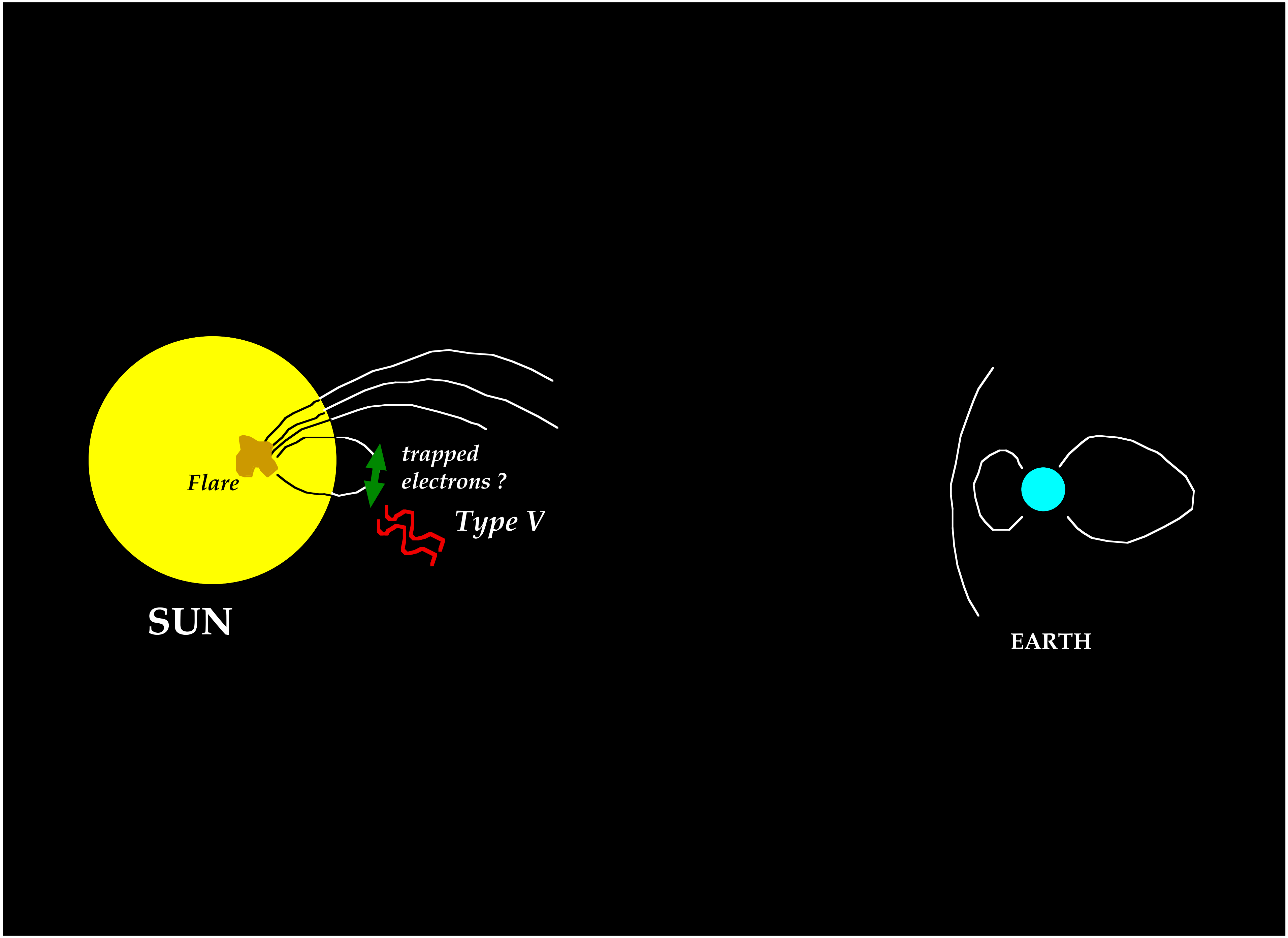
$$\nabla \times \mathbf{E}(\mathbf{r},t) + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t) = 0,$$

$$\nabla \bullet \mathbf{B}(\mathbf{r},t) = 0,$$

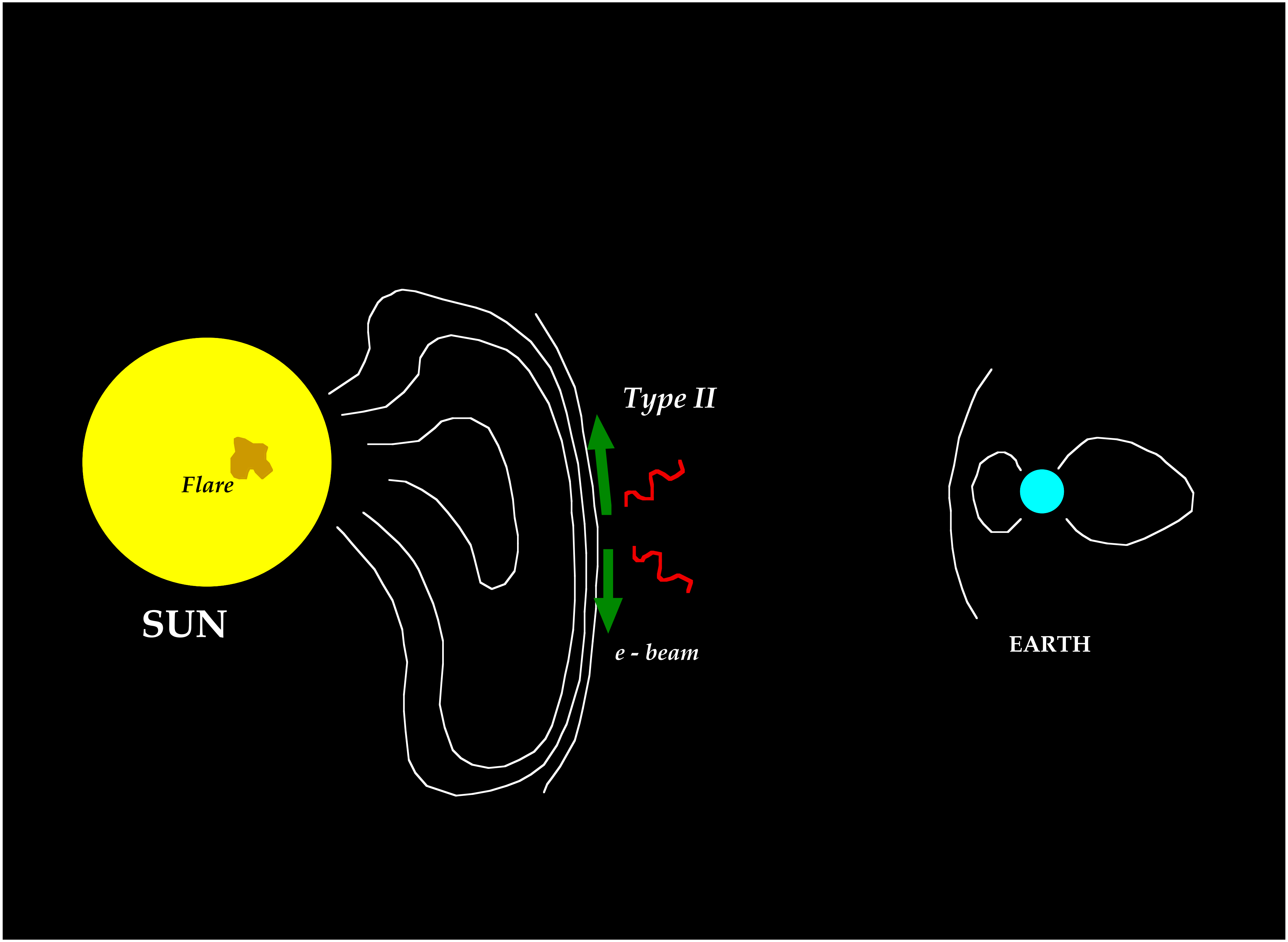
$$\nabla \bullet \mathbf{E}(\mathbf{r},t) = 4\pi \sum_a \int d\mathbf{v} N_a(\mathbf{r},\mathbf{v},t),$$

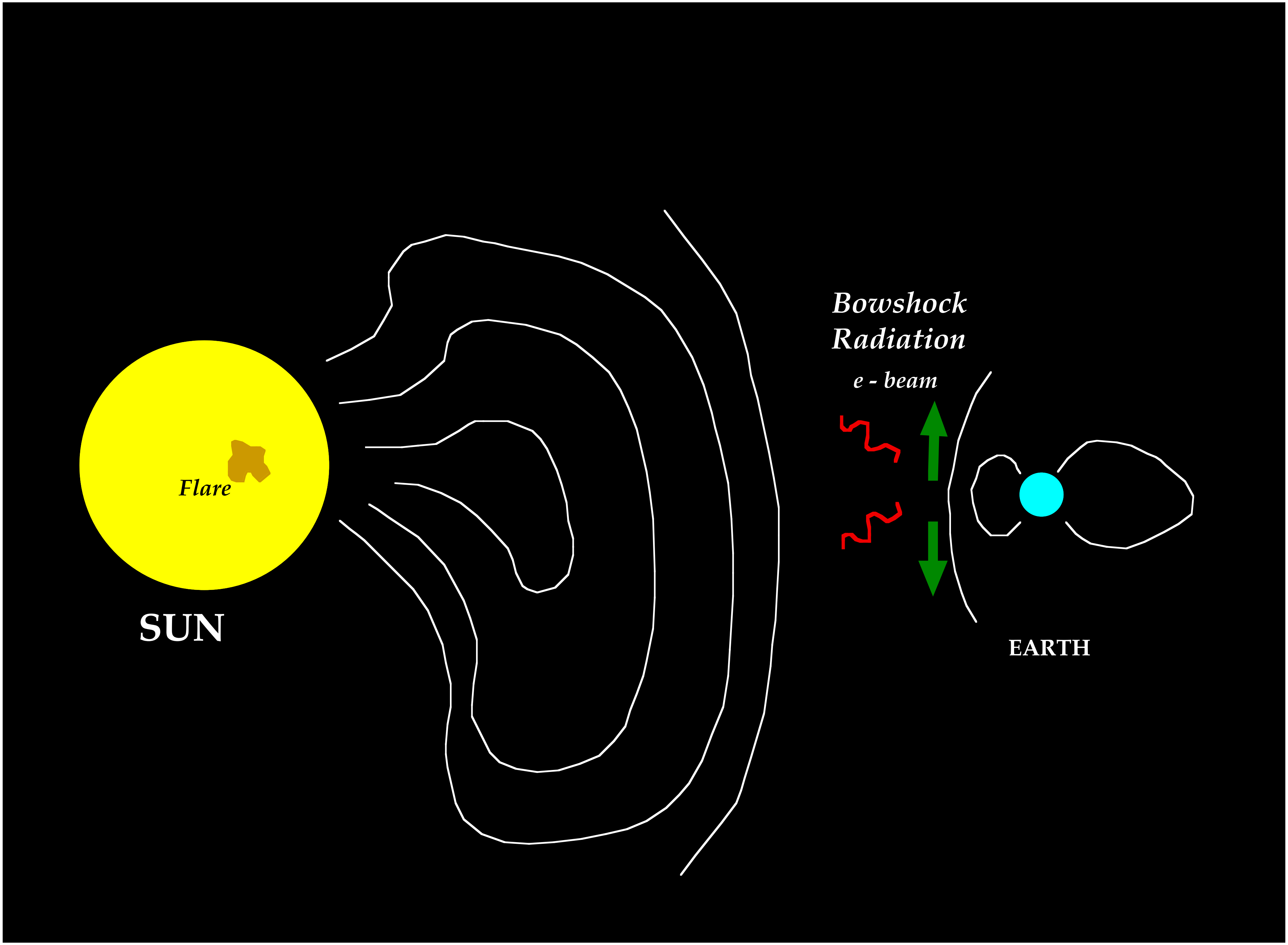
$$\nabla \times \mathbf{B}(\mathbf{r},t) - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t) = \frac{4\pi}{c}\sum_a e_a \int d\mathbf{v}\mathbf{v} N_a(\mathbf{r},\mathbf{v},t).$$


Type III burst
e - beam
Flare
SUN
EARTH

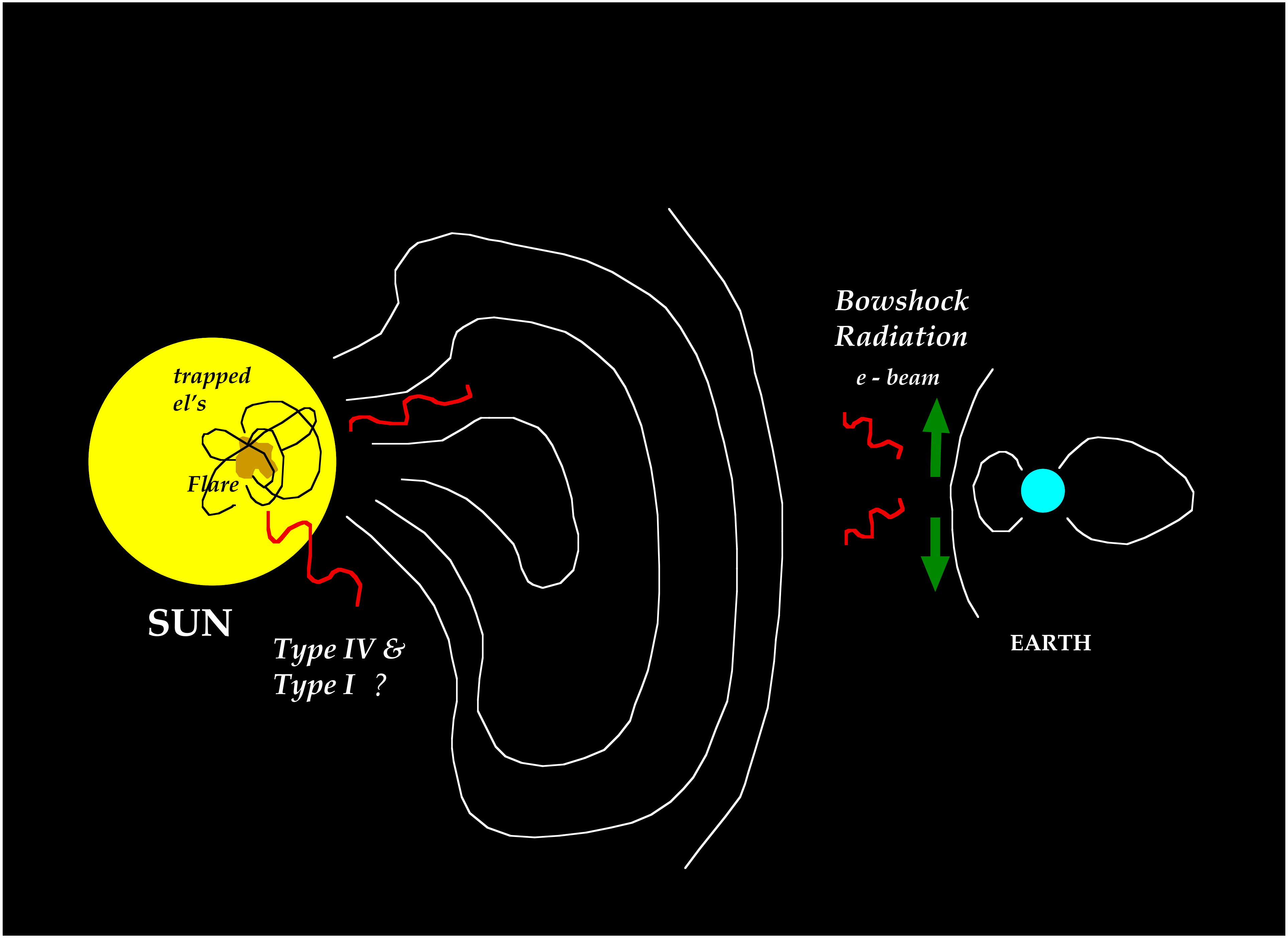


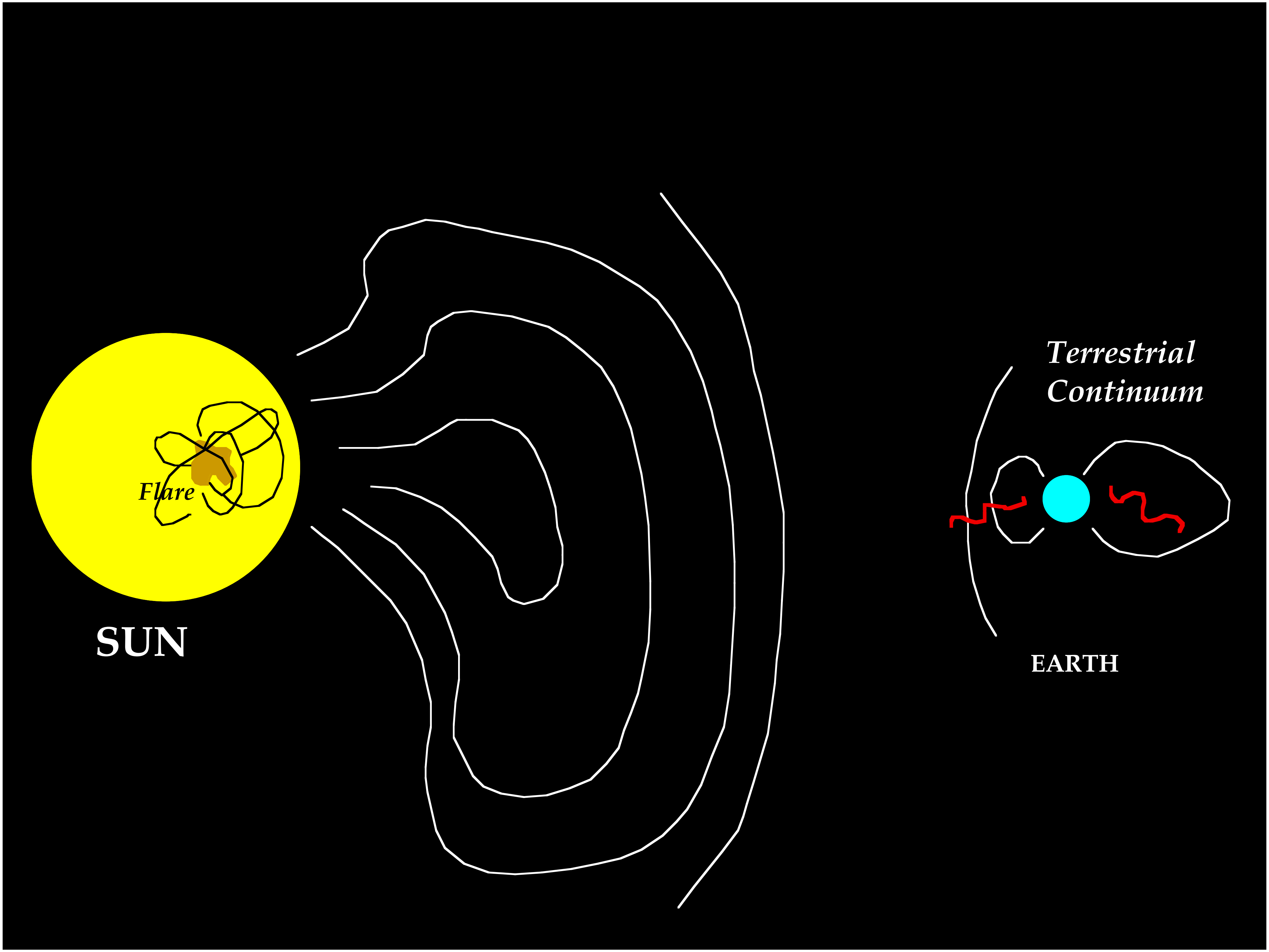
Flare
trapped
electrons ?
Type V
SUN
EARTH


Flare
SUN
Type II
e - beam
EARTH



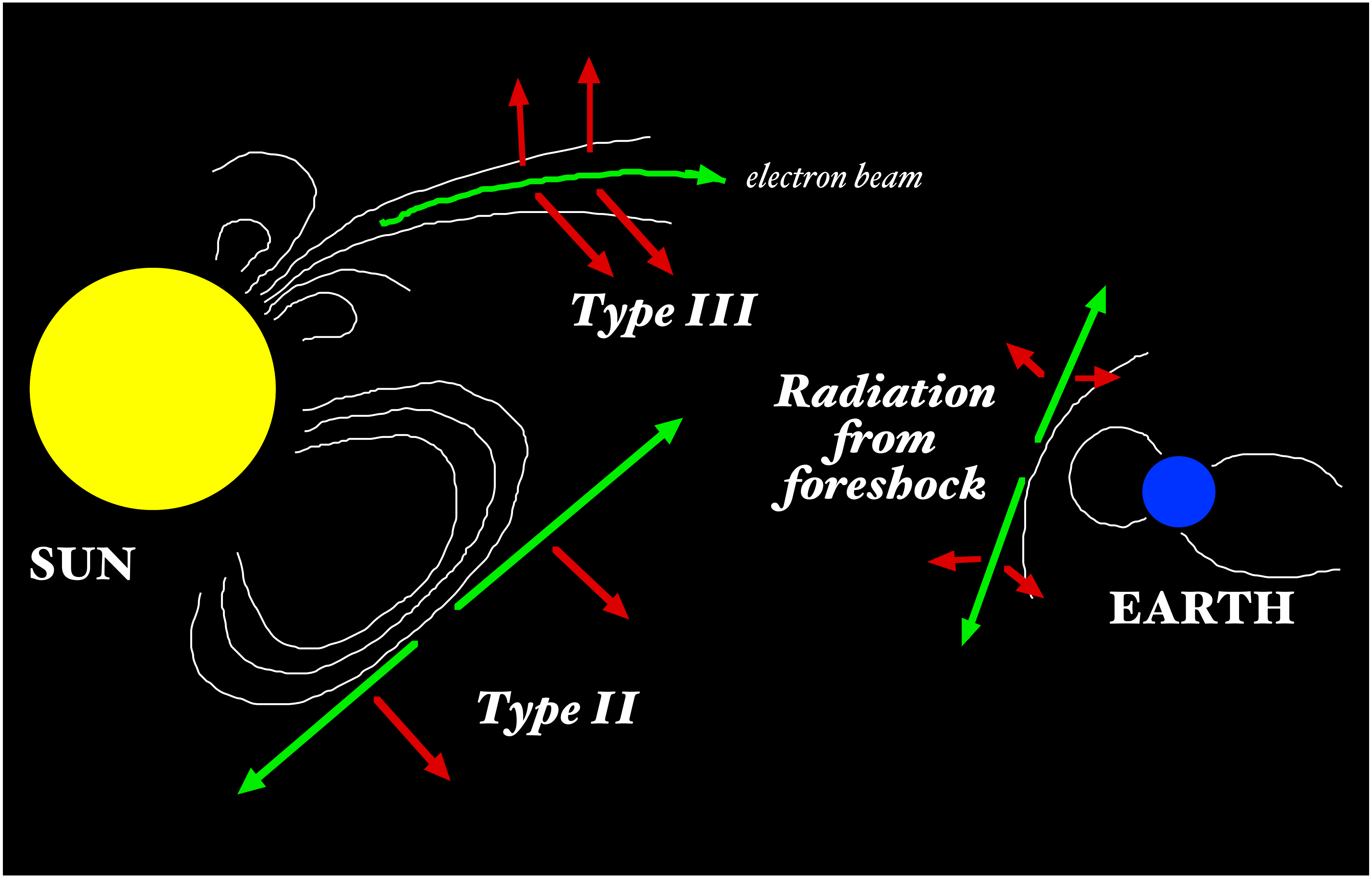
Bowshock
Radiation
e - beam
Flare
SUN
EARTH

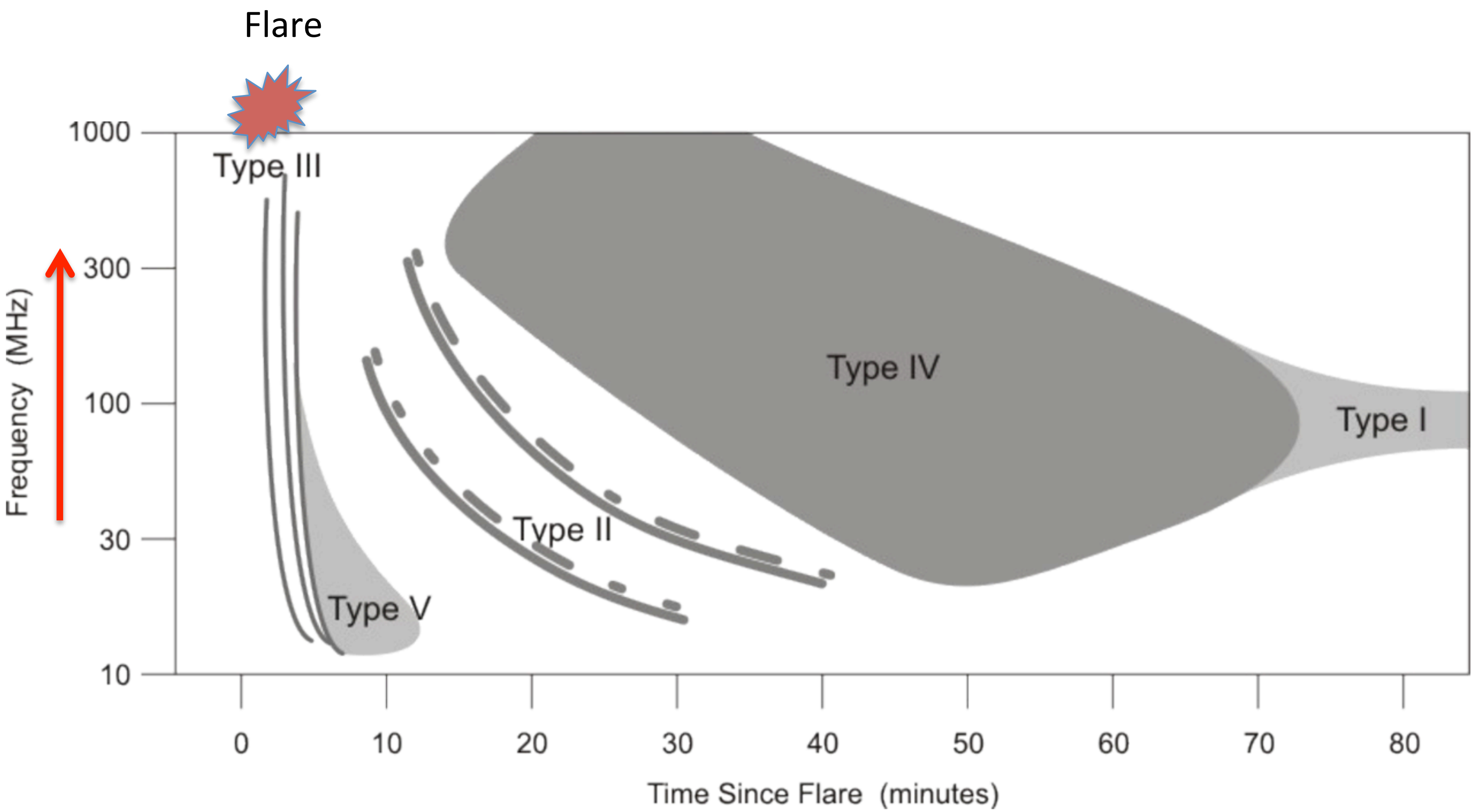

trapped
el's
Flare
SUN
Type IV &
Type I ?
Bowshock
Radiation
e - beam
EARTH



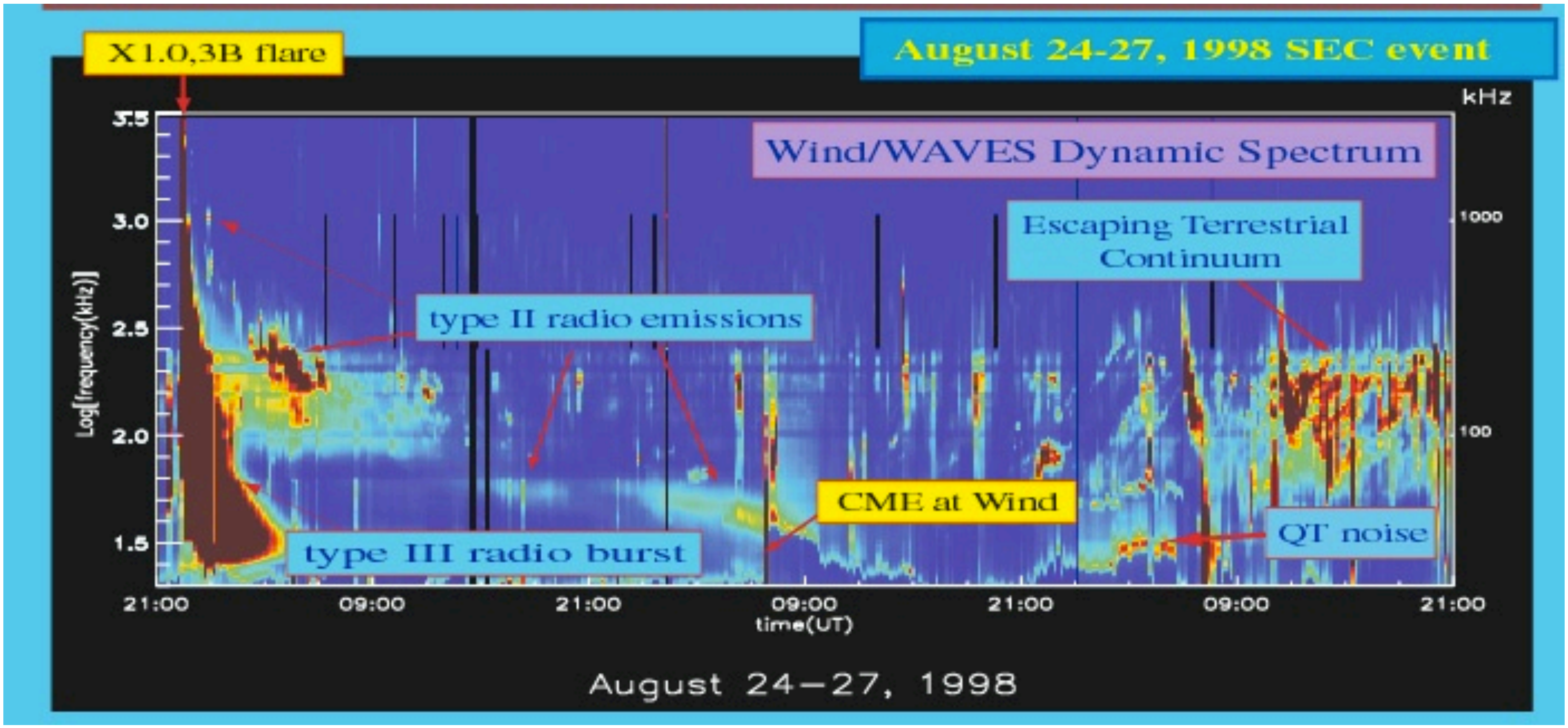
Flare
SUN
Terrestrial
Continuum
EARTH

Radiations in Space



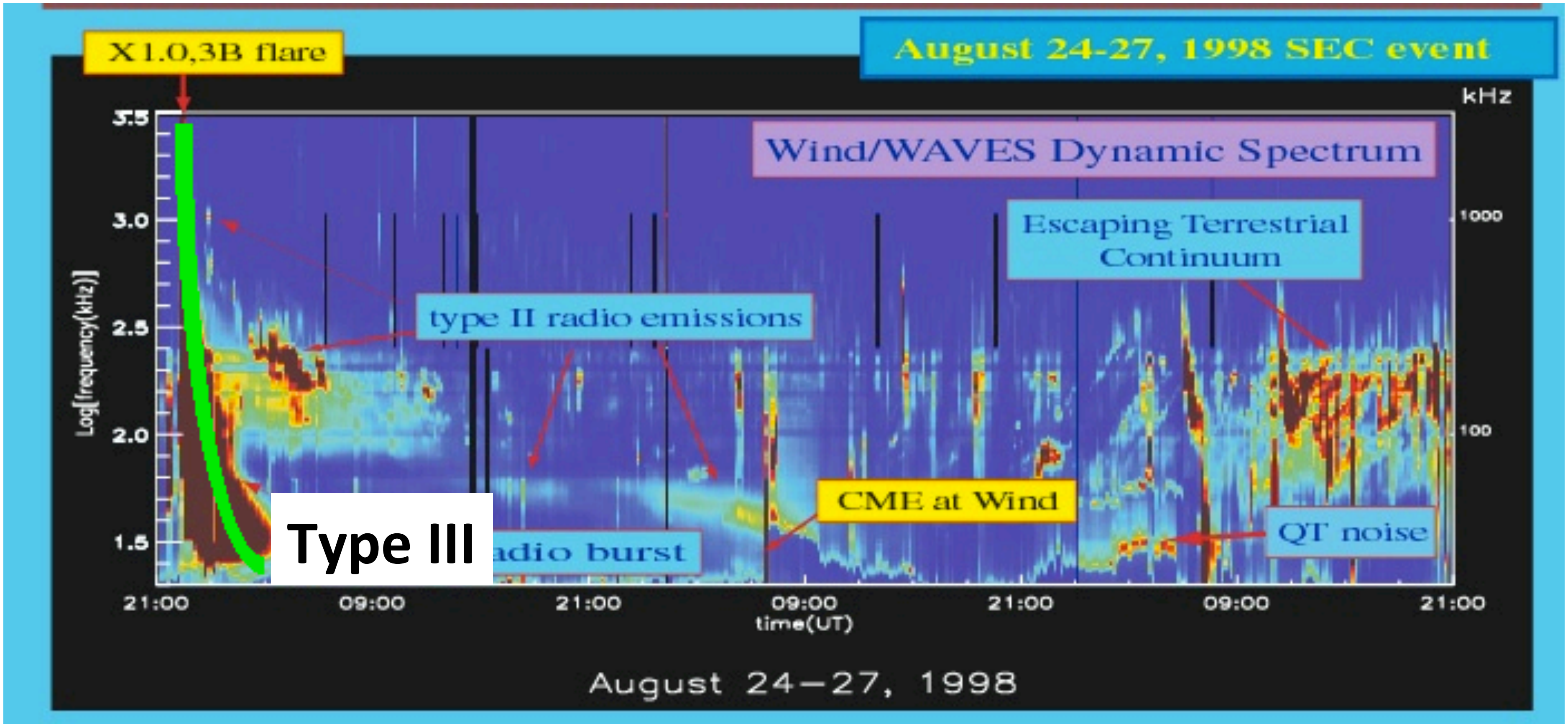


Flare
1000
300
100
30
10
Frequency (MHz)
Type III
Type IV
Type I
Type II
Type V
0
10
20
30
40
50
60
70
80
Time Since Flare (minutes)

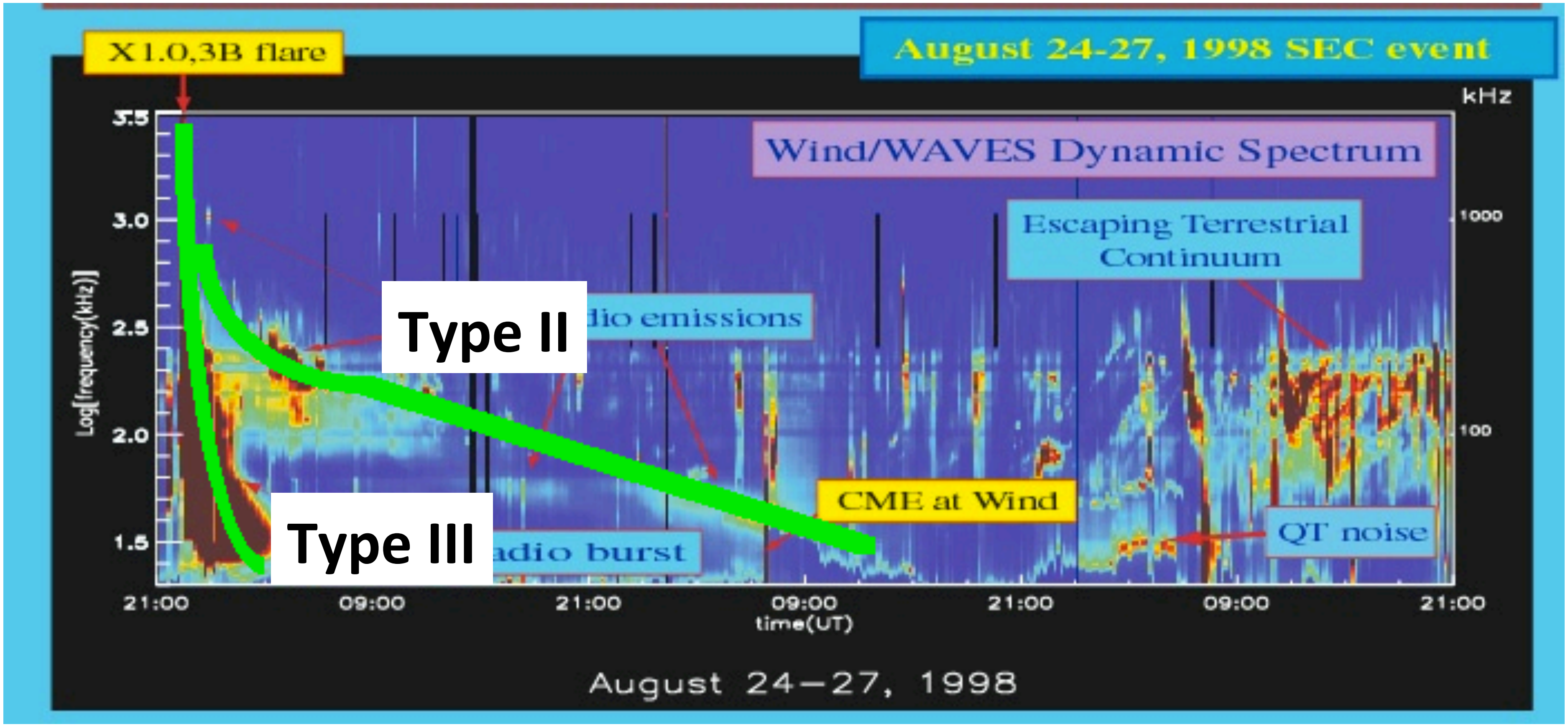


X1.0,3B flare
August 24-27, 1998 SEC event
kHz
Wind/WAVES Dynamic Spectrum
Escaping Terrestrial Continuum
type II radio emissions
CME at Wind
QT noise
type III radio burst
Log[frequency(kHz)]
3.5
3.0
2.5
2.0
1.5
1000
100
21:00
09:00
21:00
09:00
21:00
09:00
21:00
time(UT)
August 24–27, 1998

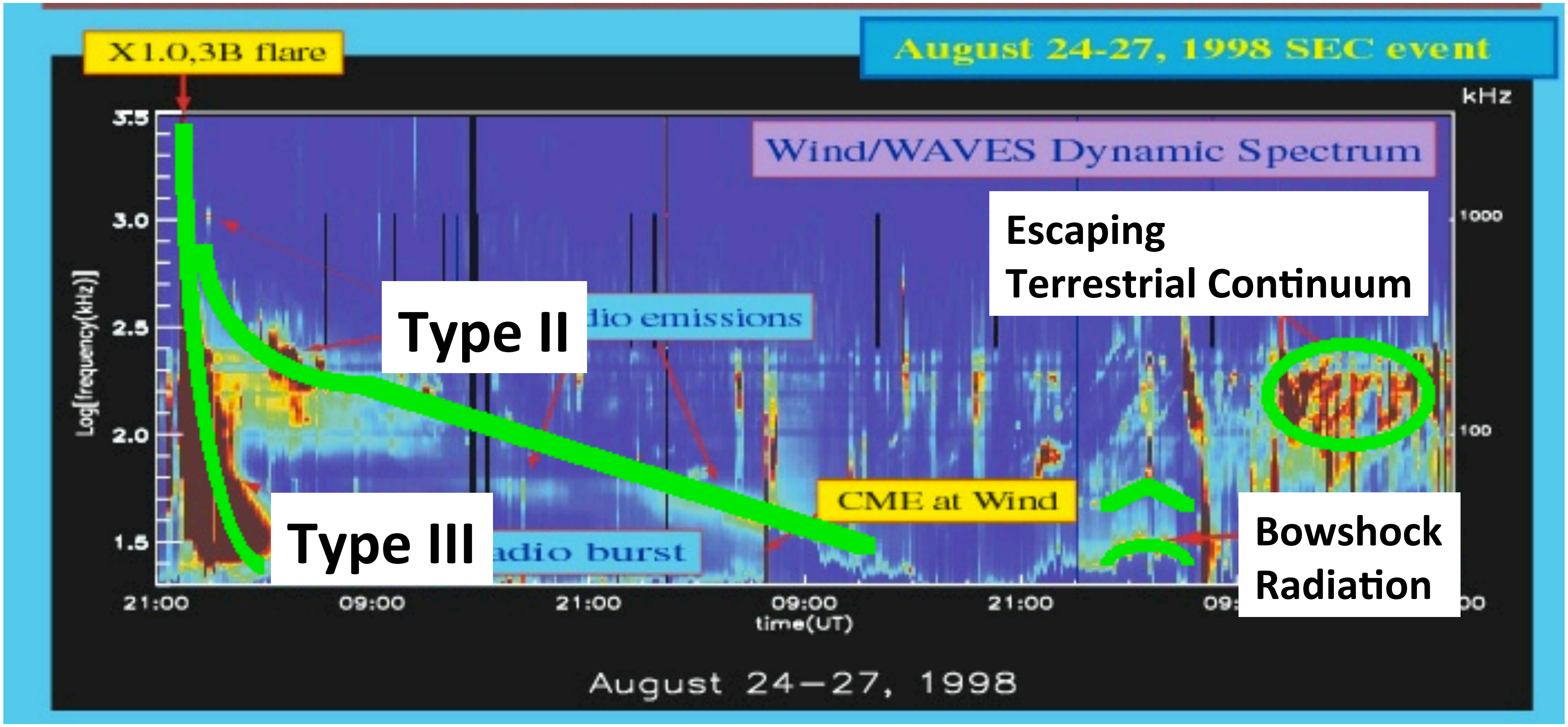
Reiner et al.


X1.0,3B flare
August 24-27, 1998 SEC event
kHz
Wind/WAVES Dynamic Spectrum
Escaping Terrestrial Continuum
type II radio emissions
Type III
adio burst
CME at Wind
QT noise
Log[frequency(kHz)]
3.5
3.0
2.5
2.0
1.5
1000
100
21:00
09:00
21:00
09:00
21:00
09:00
21:00
time(UT)
August 24–27, 1998

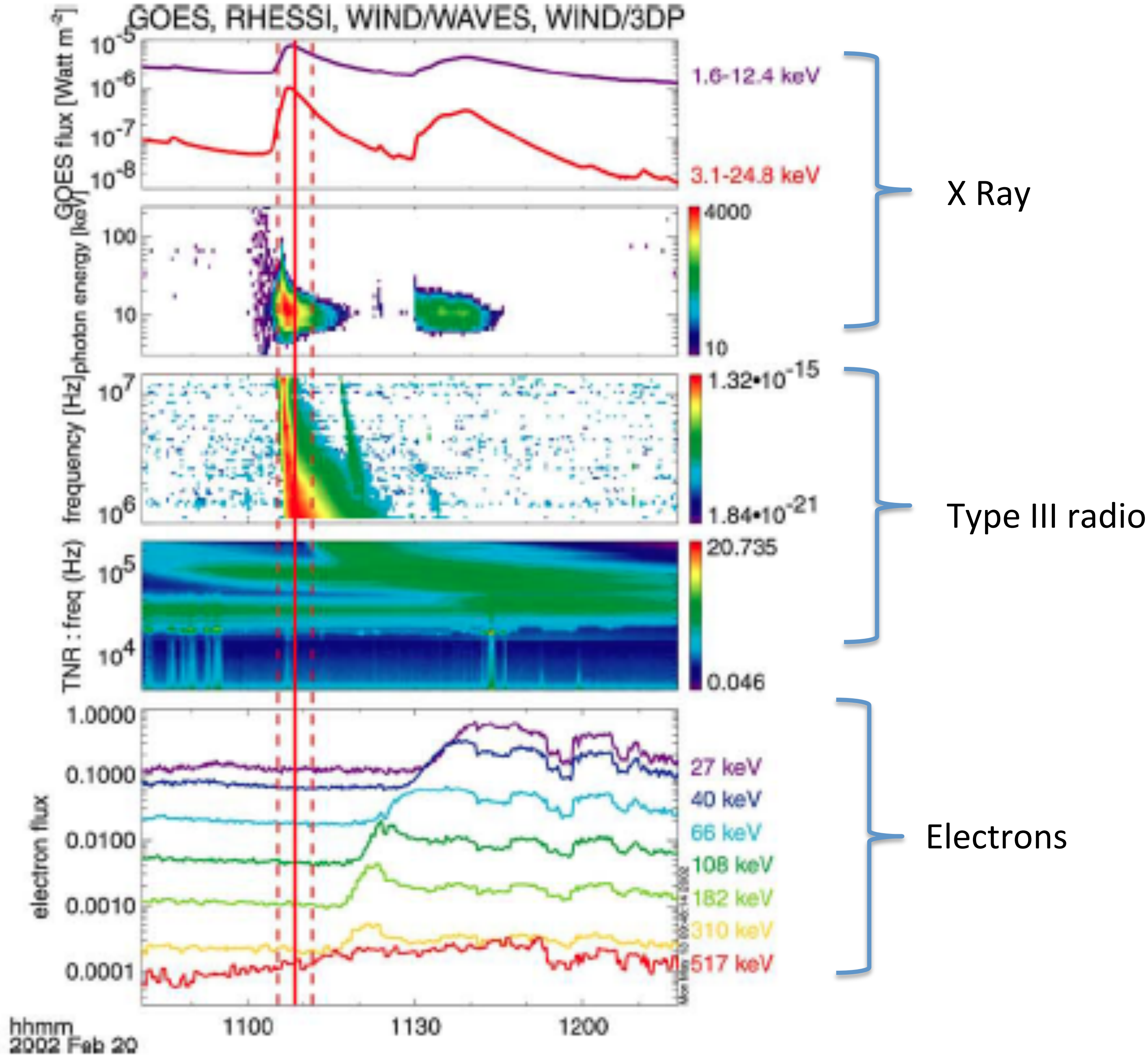
Reiner et al.


X1.0,3B flare
August 24-27, 1998 SEC event
Wind/WAVES Dynamic Spectrum
Escaping Terrestrial Continuum
Type II
Type III
CME at Wind
QT noise
Log[frequency(kHz)]
kHz
time(UT)
August 24–27, 1998

Reiner et al.


X1.0,3B flare
August 24-27, 1998 SEC event
Wind/WAVES Dynamic Spectrum
kHz
Escaping
Terrestrial Continuum
Type II
dio emissions
Type III
adio burst
CME at Wind
Bowshock
Radiation
3.5
3.0
2.5
2.0
1.5
Log[frequency(kHz)]
1000
100
21:00
09:00
21:00
09:00
time(UT)
21:00
August 24–27, 1998

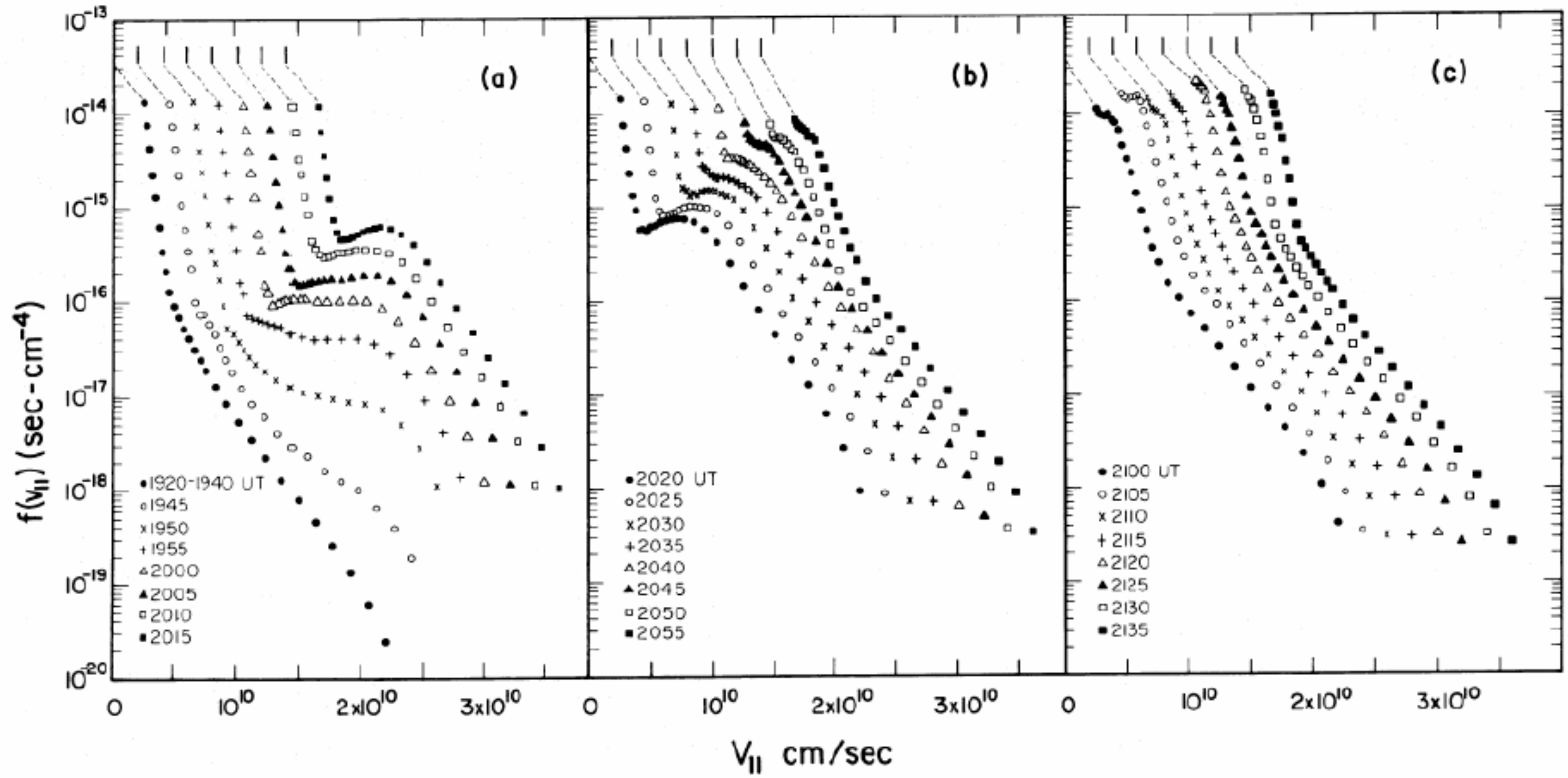
Reiner et al.

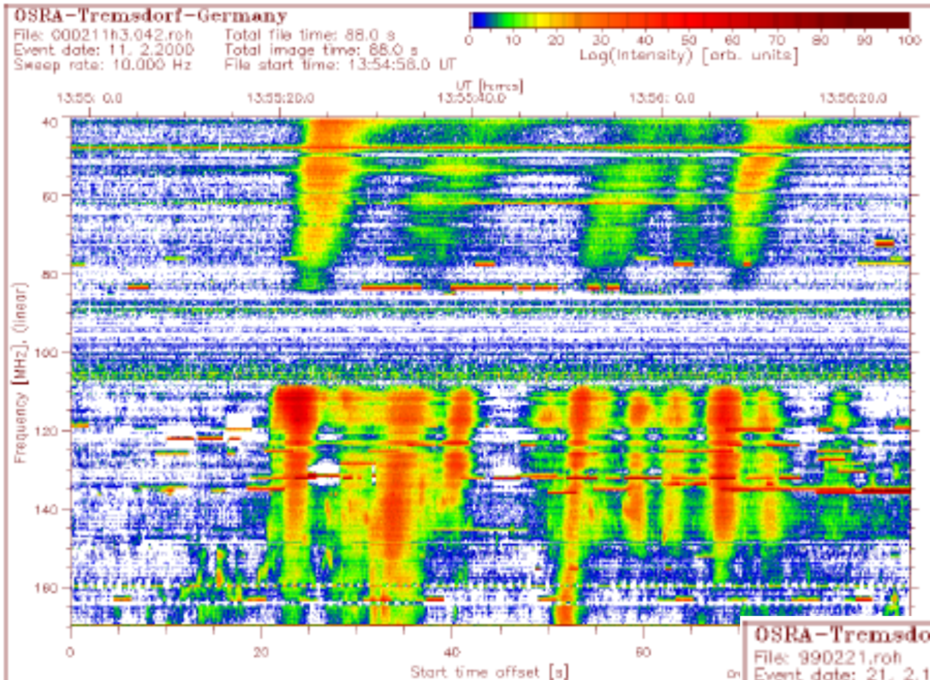


GOES, RHESSI, WIND/WAVES, WIND/3DP
GOES flux [Watt m^{-2}]
1.6-12.4 keV
3.1-24.8 keV
photon energy [keV]
4000
10
frequency [Hz]
$1.32 \cdot 10^{-15}$
$1.84 \cdot 10^{-21}$
TNR : freq (Hz)
20.735
0.046
electron flux
27 keV
40 keV
66 keV
108 keV
182 keV
310 keV
517 keV
hhmm
2002 Feb 20
1100
1130
1200
X Ray
Type III radio
Electrons

ISEE-3 type III
1979 Feb 17

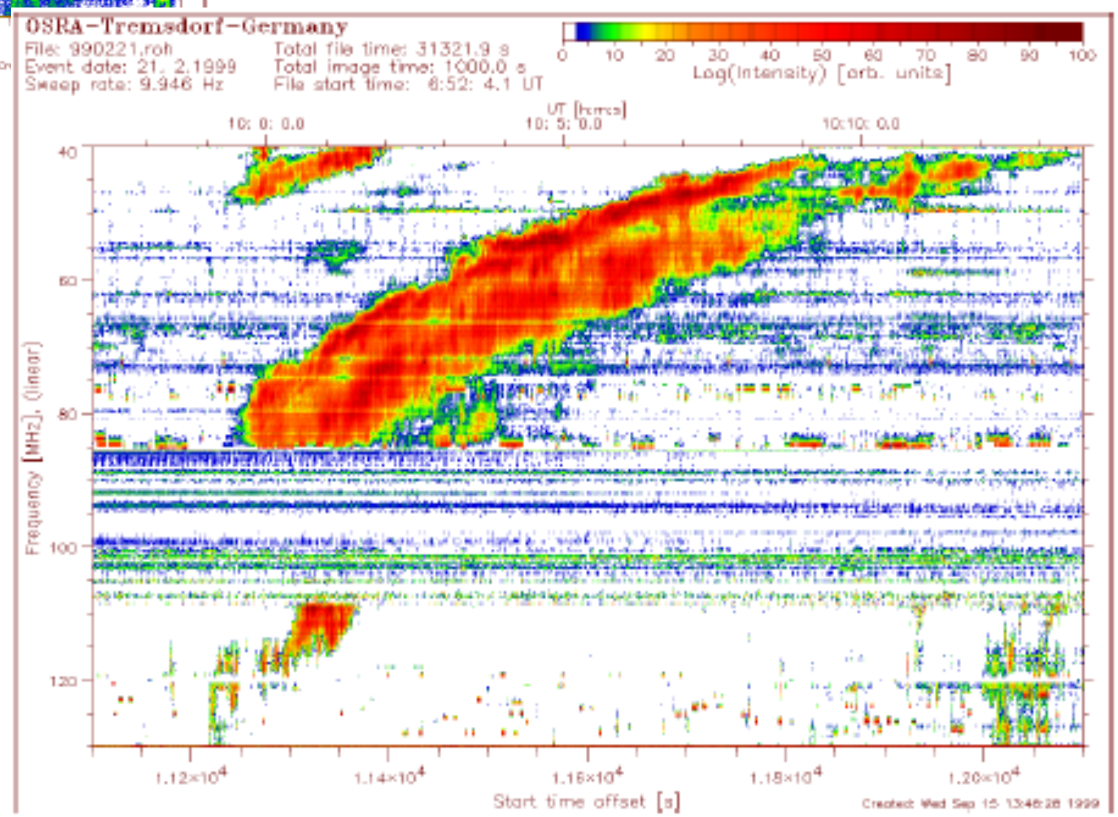
Lin 1981

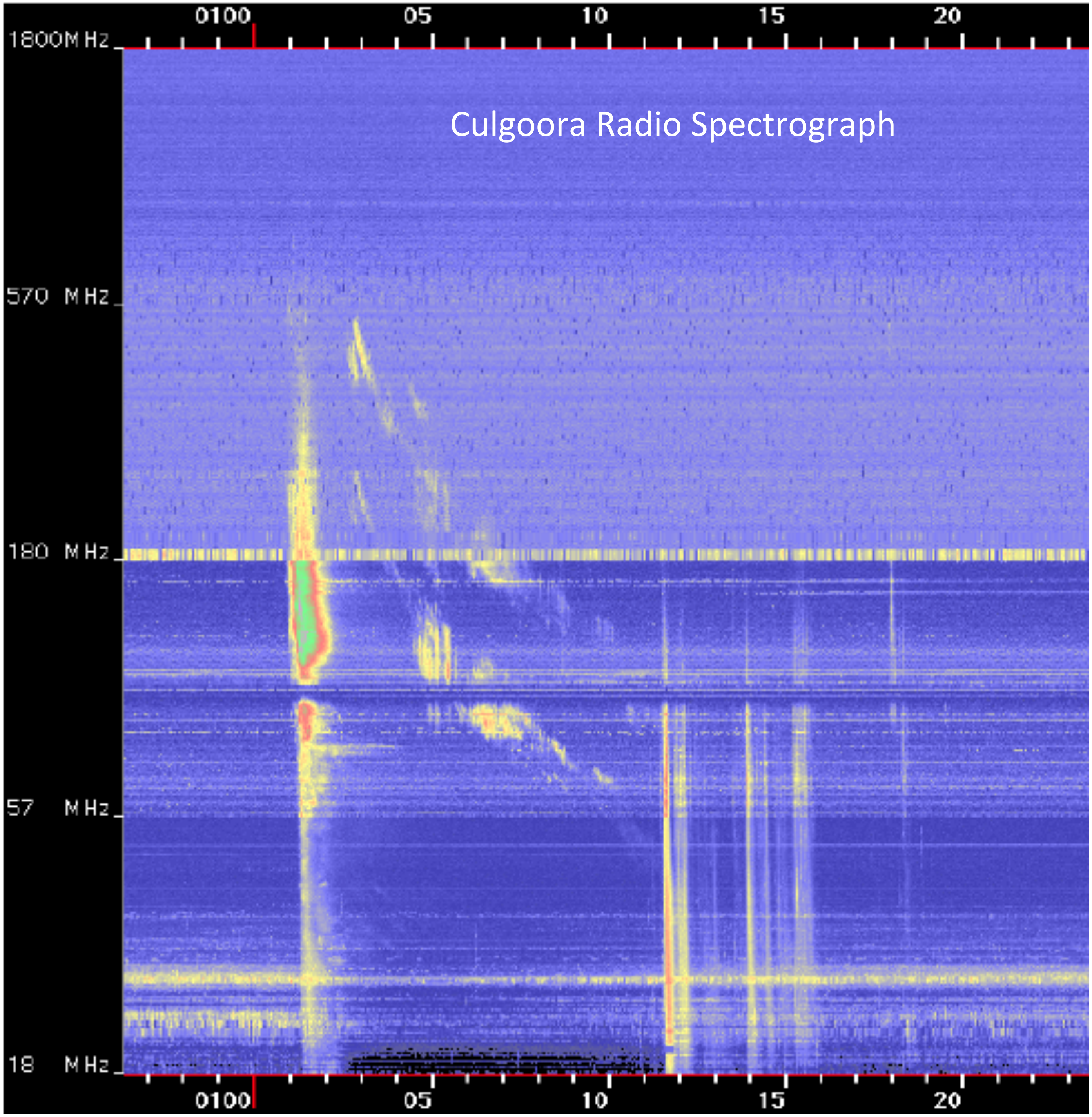




Most type III bursts do not have the iconic F-H pair emission structure

Type II emissions are somewhat more clearly identifiable with the beam-induced F-H pair emissions



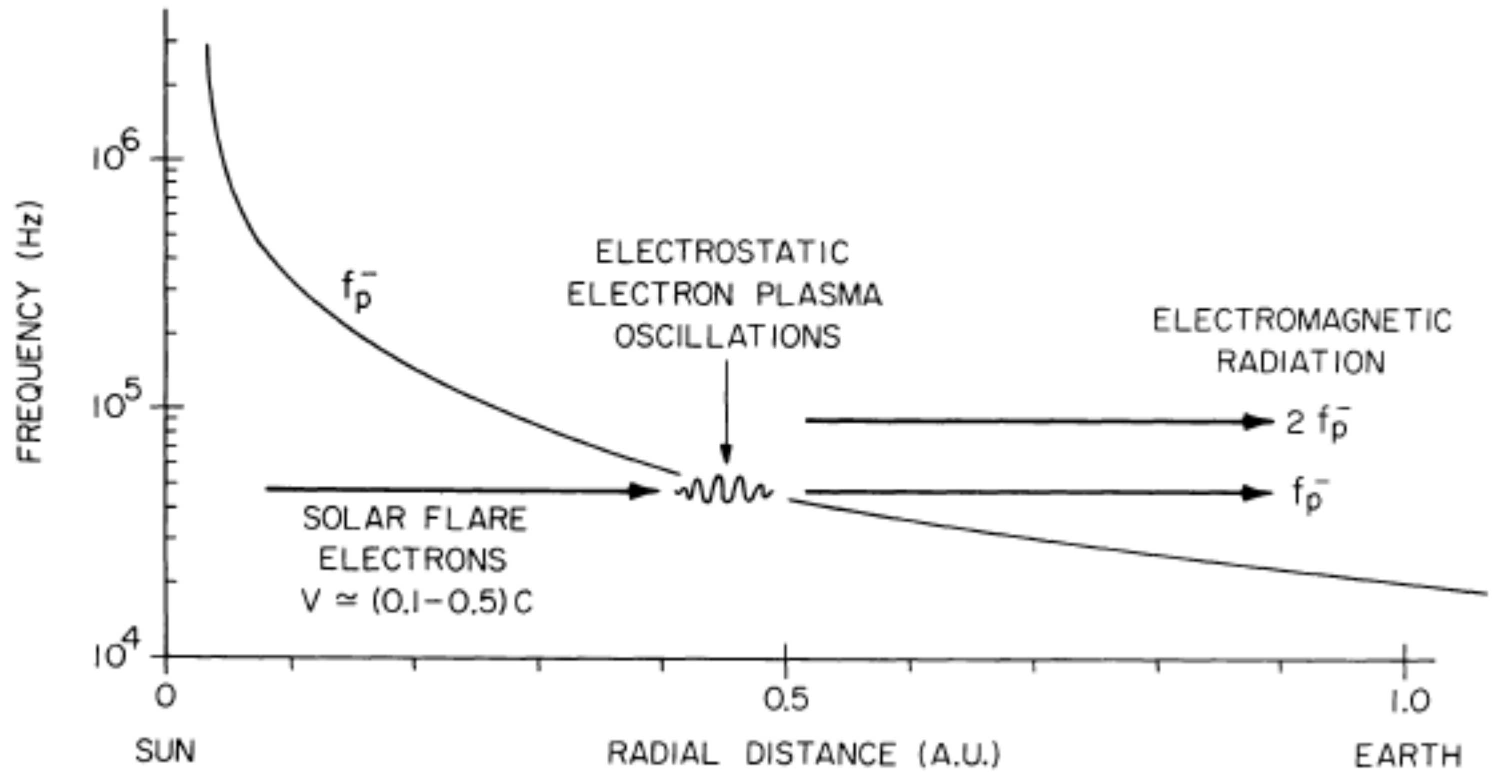


Culgoora Radio Spectrograph
1800MHz
570 MHz
180 MHz
57 MHz
18 MHz
0100
05
10
15
20

Plasma emission scenario

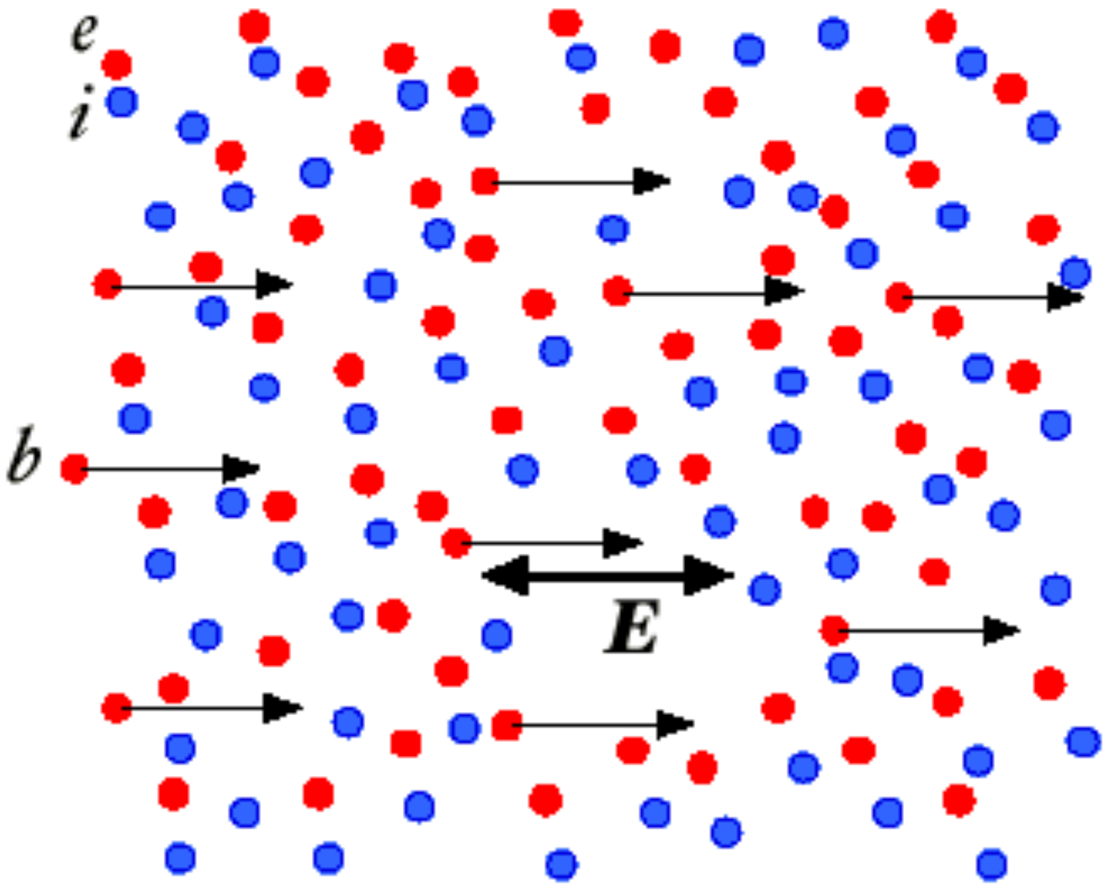
(Ginzburg & Zeleznyakov, 1958; Melrose, 1970s; Robinson, Cairns, 1980 & 1990s)

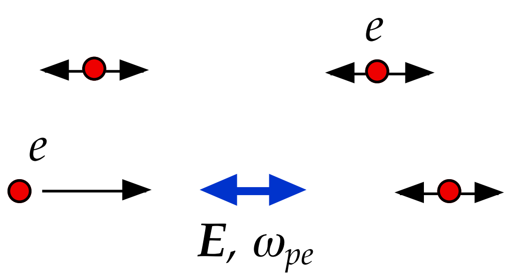
- Electron **beam** produced during flare
- Generation of **Langmuir** waves
- **Backscattered** Langmuir waves by nonlinear processes
- **Harmonic** emission by merging of Langmuir waves
- **Fundamental** emission by Langmuir wave decay



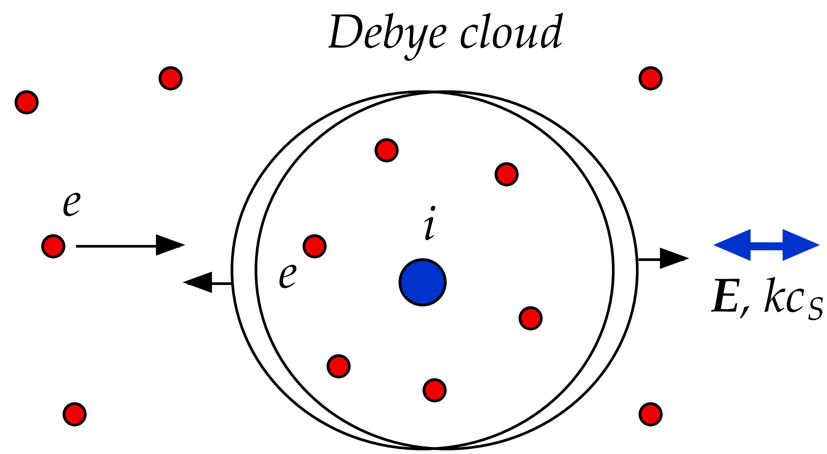
FREQUENCY (Hz)
10^6
10^5
10^4
f_p^-
ELECTROSTATIC
ELECTRON PLASMA
OSCILLATIONS
ELECTROMAGNETIC
RADIATION
$2 f_p^-$
f_p^-
SOLAR FLARE
ELECTRONS
V ≃ (0.1–0.5) C
0
0.5
1.0
SUN
RADIAL DISTANCE (A.U.)
EARTH

Beam-generated Langmuir turbulence





Langmuir oscillation

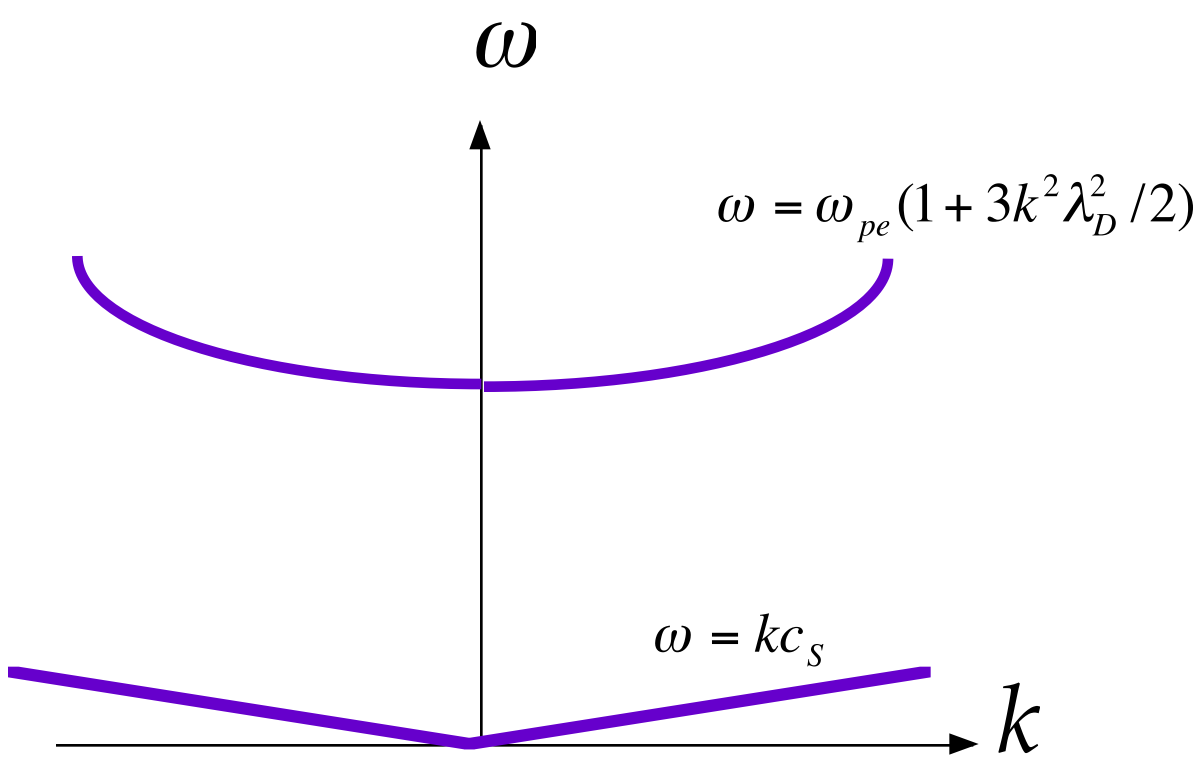


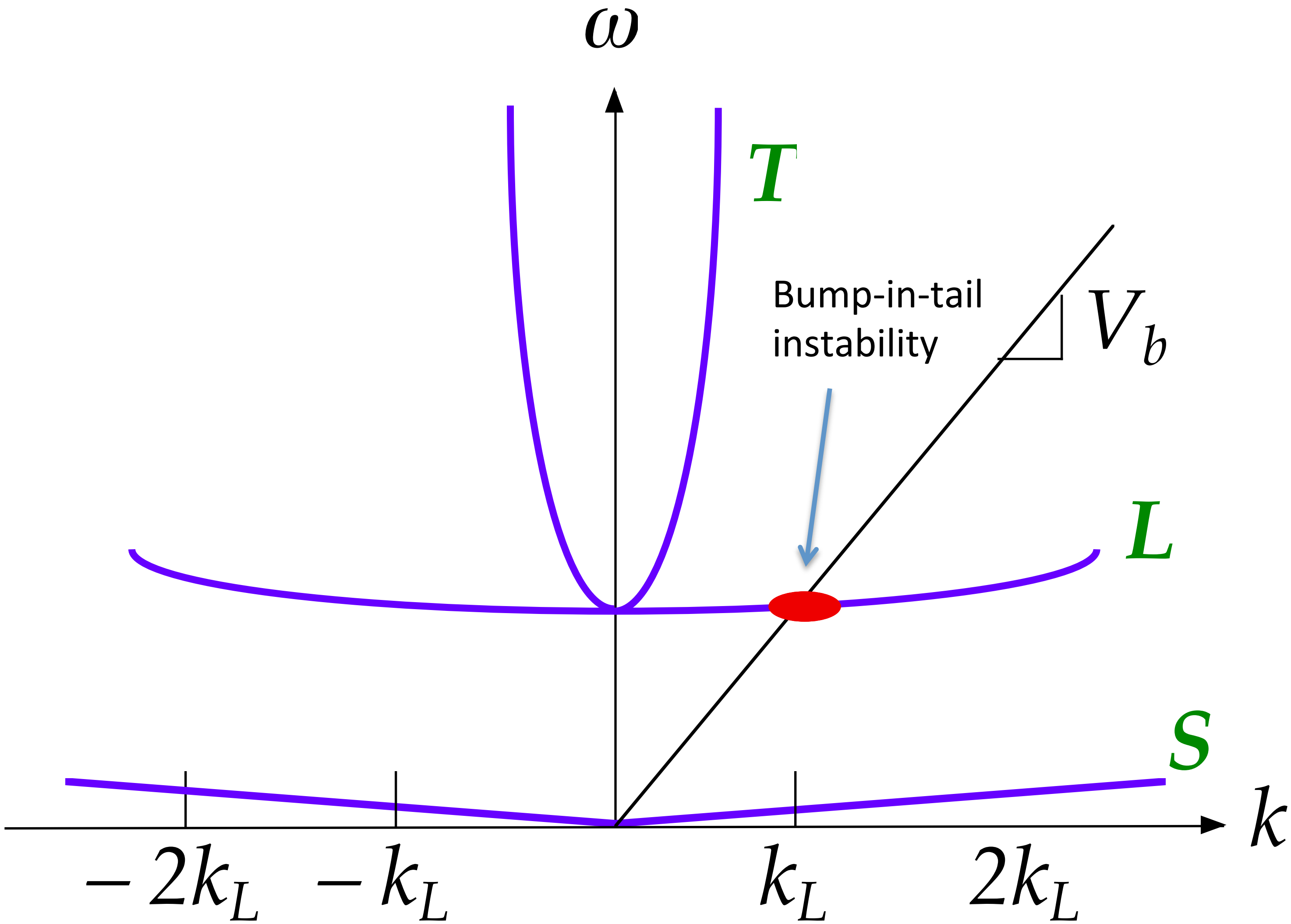
Ion-sound wave

$$\mathbf{E}(\mathbf{x},t) = \mathbf{E}\cos(\mathbf{k} \bullet \mathbf{x} - \omega t),$$

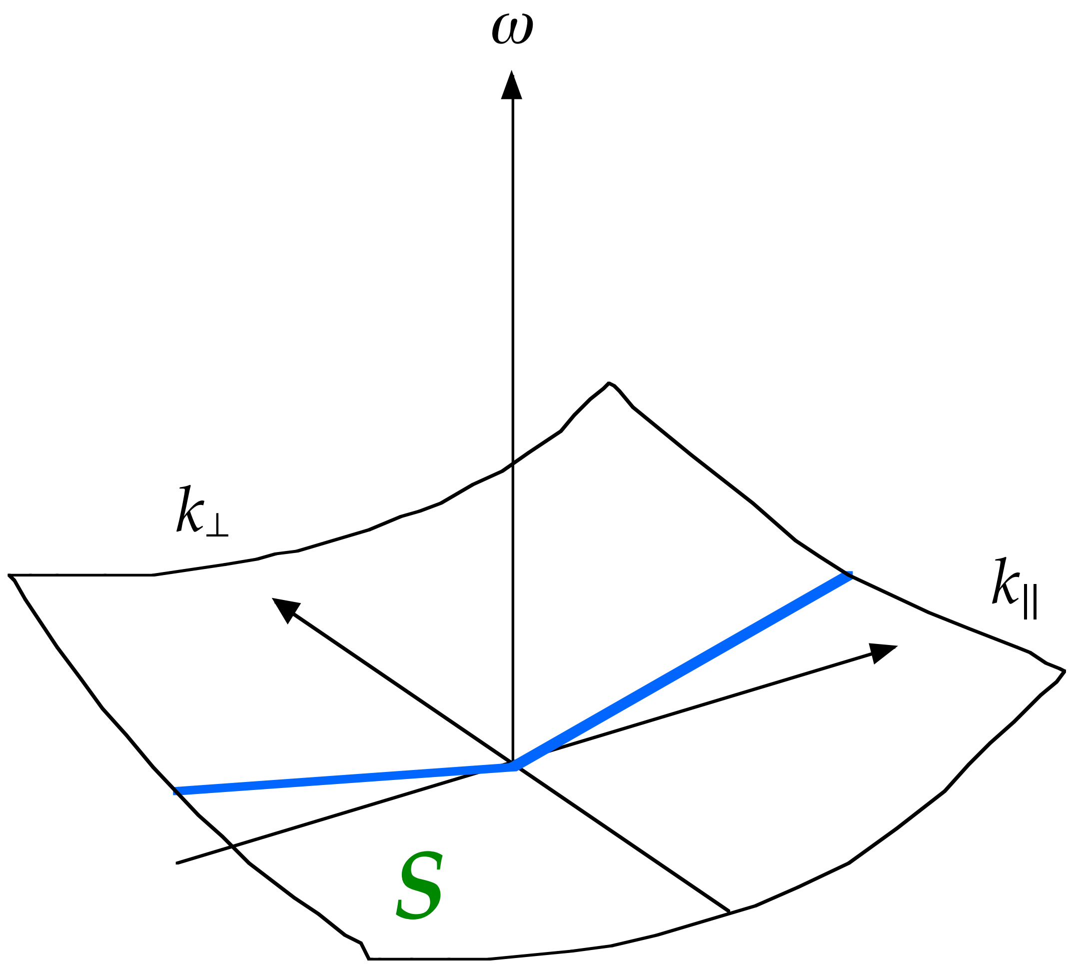
$$\omega = \omega_{pe}\left(1 + \frac{3}{2}k^2\lambda_D^2\right) = \sqrt{\frac{4\pi ne^2}{m_e}}\left(1 + k^2\frac{3T_e}{4\pi ne^2}\right), \quad or$$

$$\omega = kc_S = k\sqrt{\frac{T_e}{m_i}}.$$

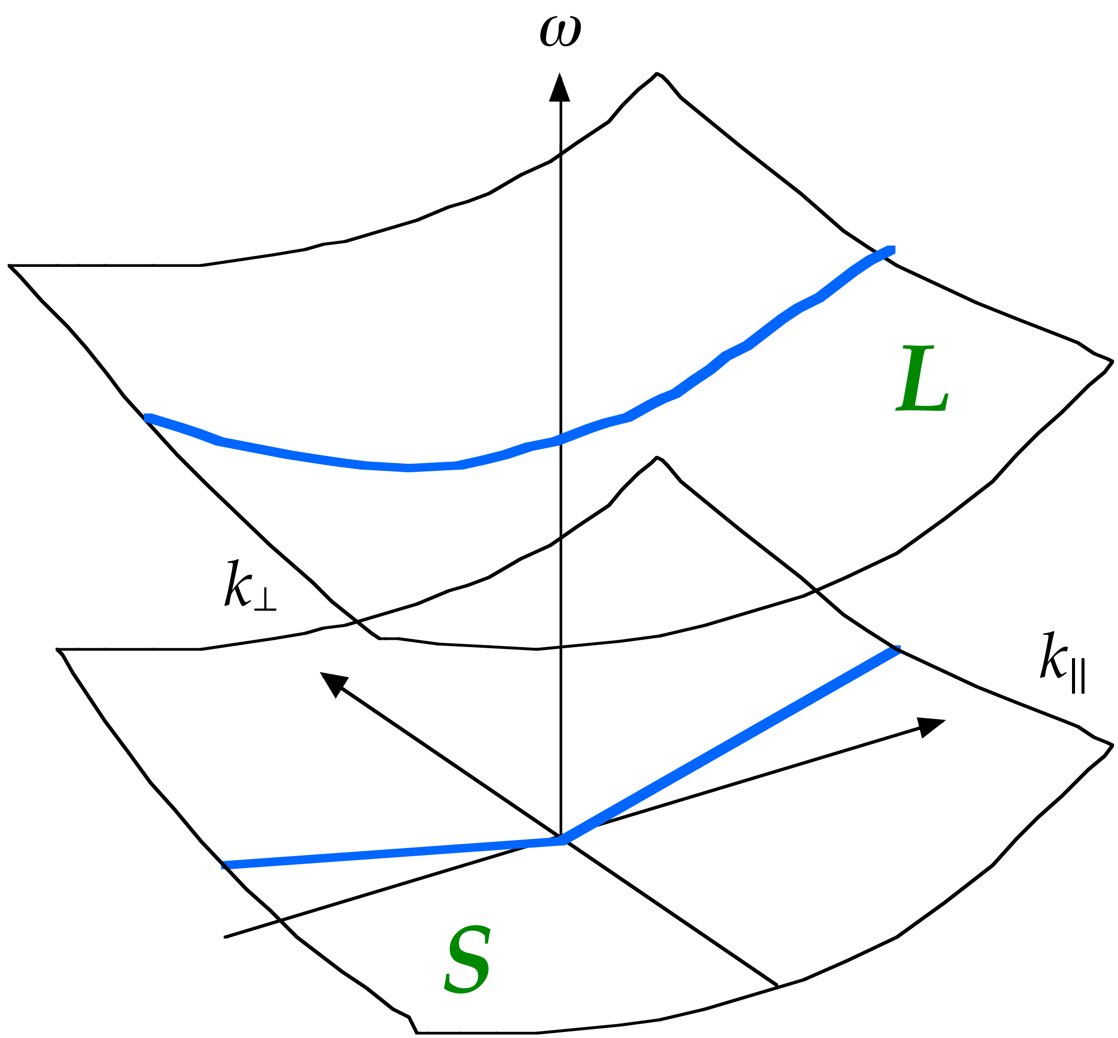




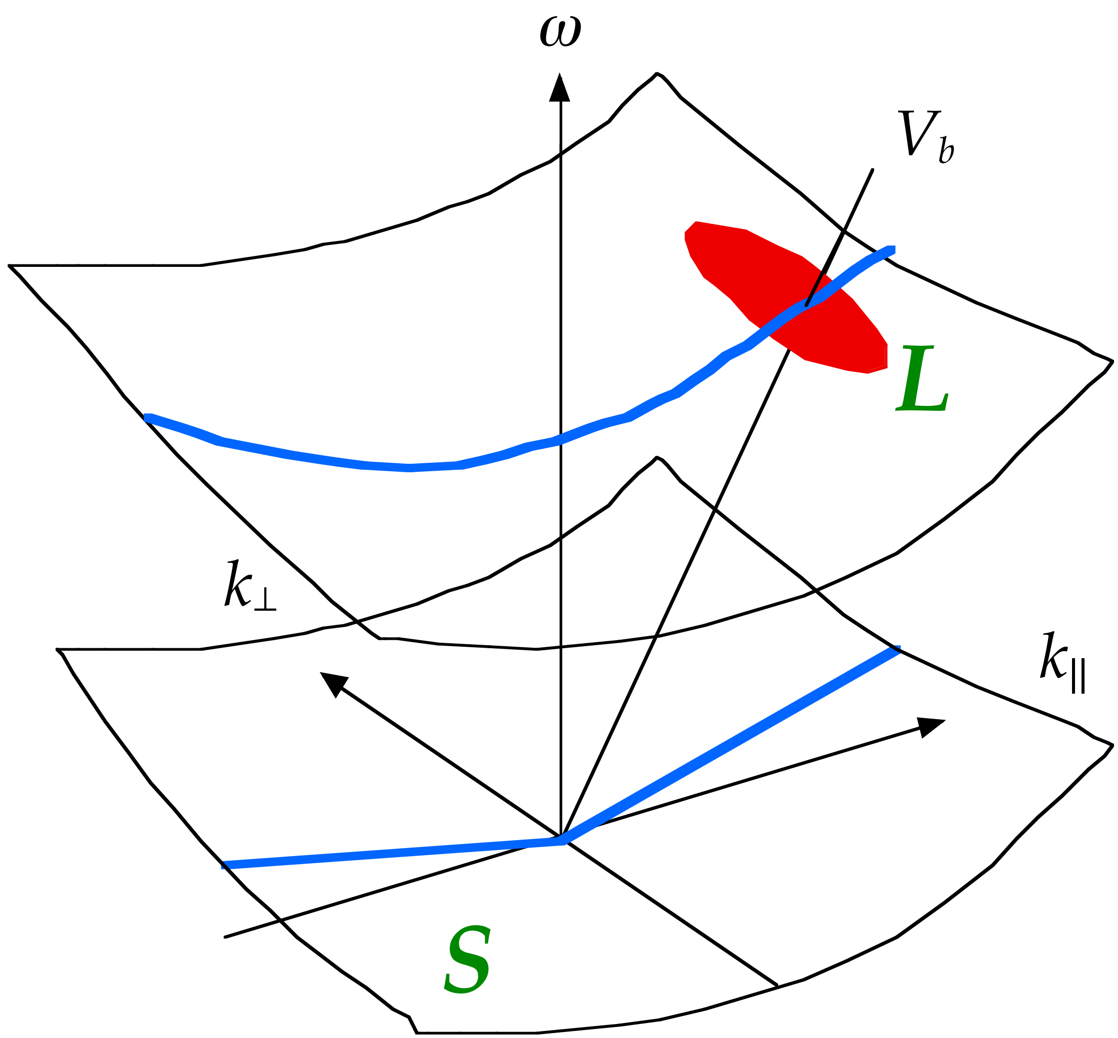
ω
T
Bump-in-tail
instability
V_b
L
S
k
$-2k_L$
$-k_L$
k_L
$2k_L$



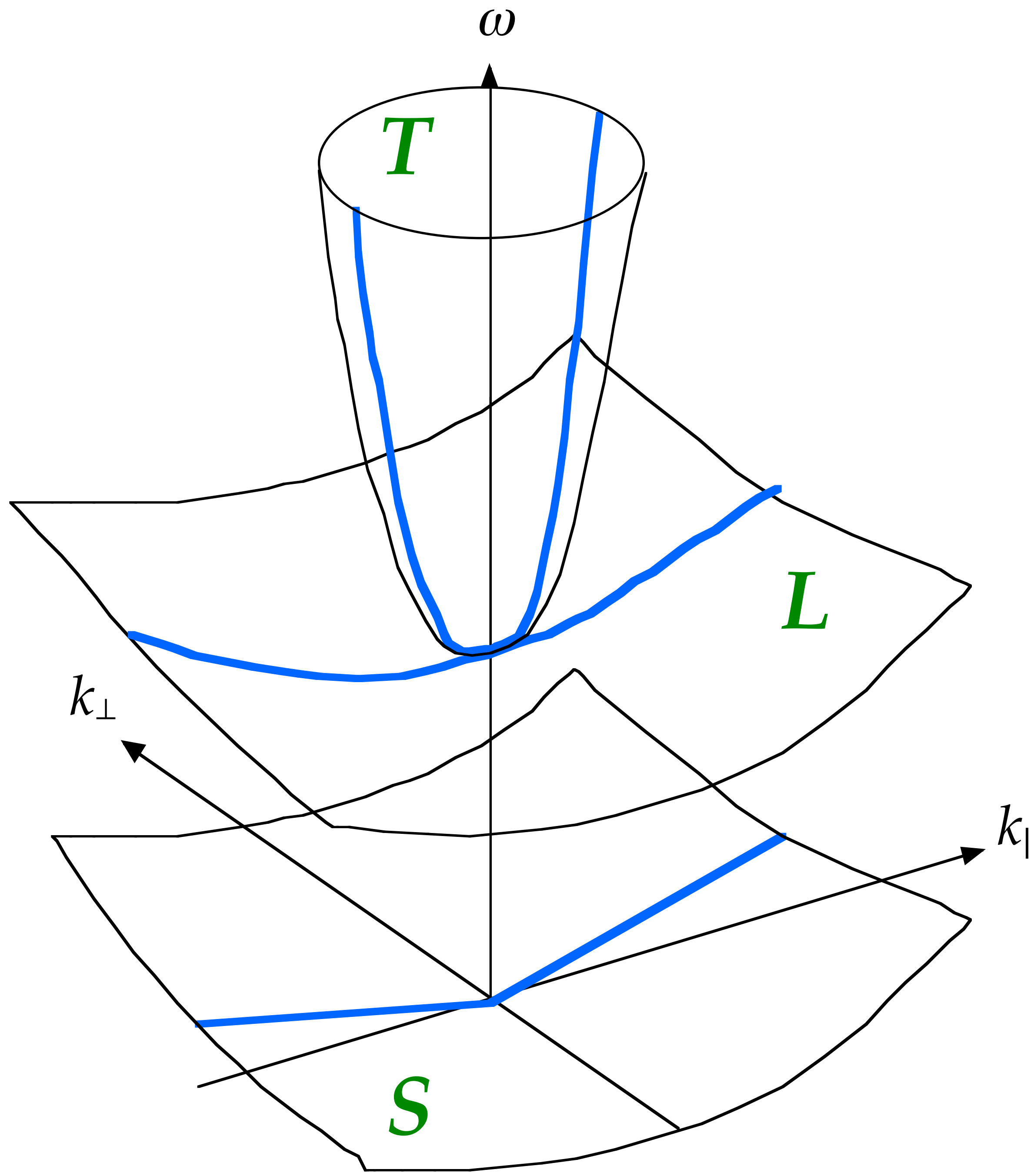
ω
k⊥
k∥
S



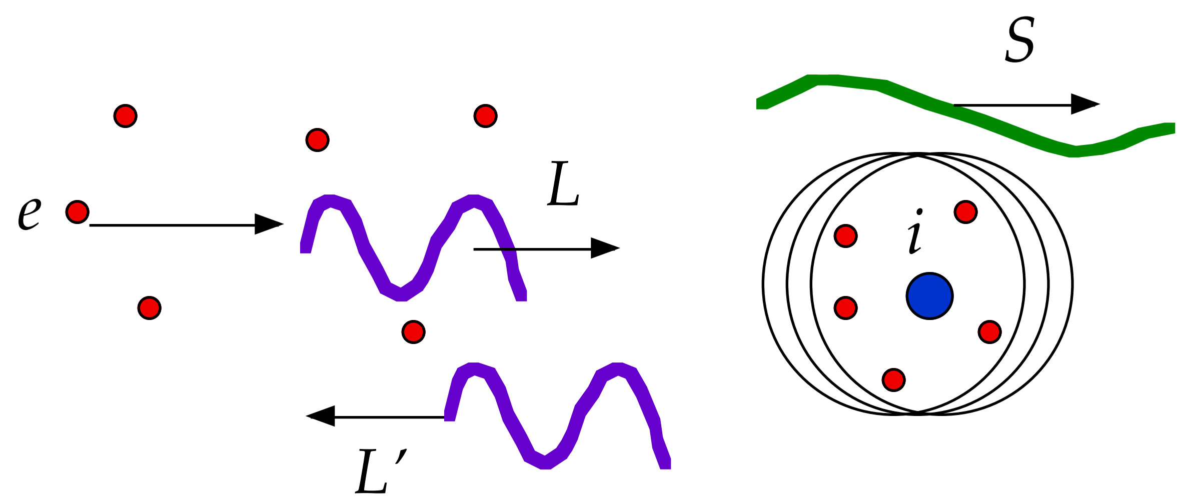
ω
L
k⊥
k∥
S

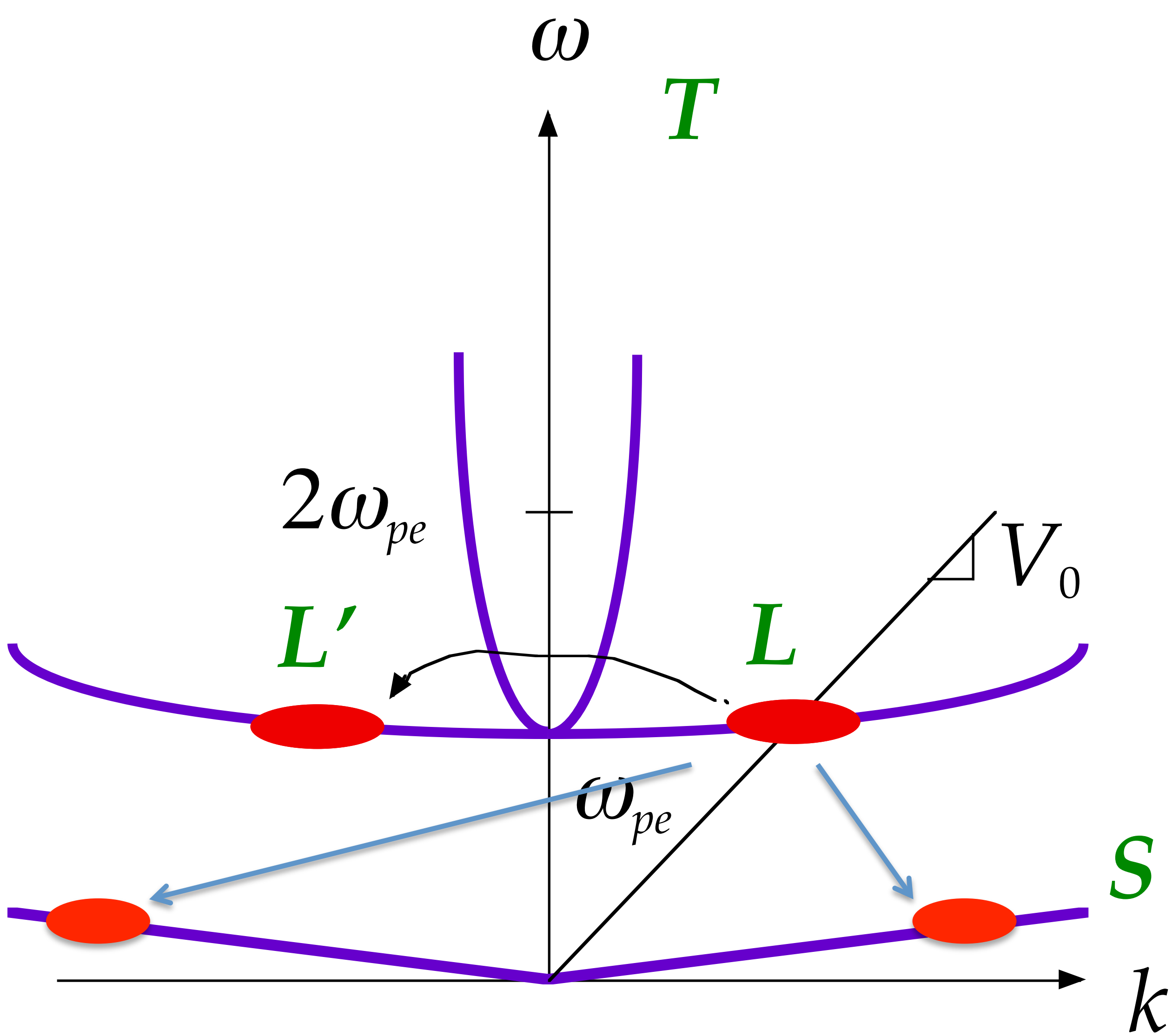


ω
V_b
L
$k_\perp$
$k_\parallel$
S

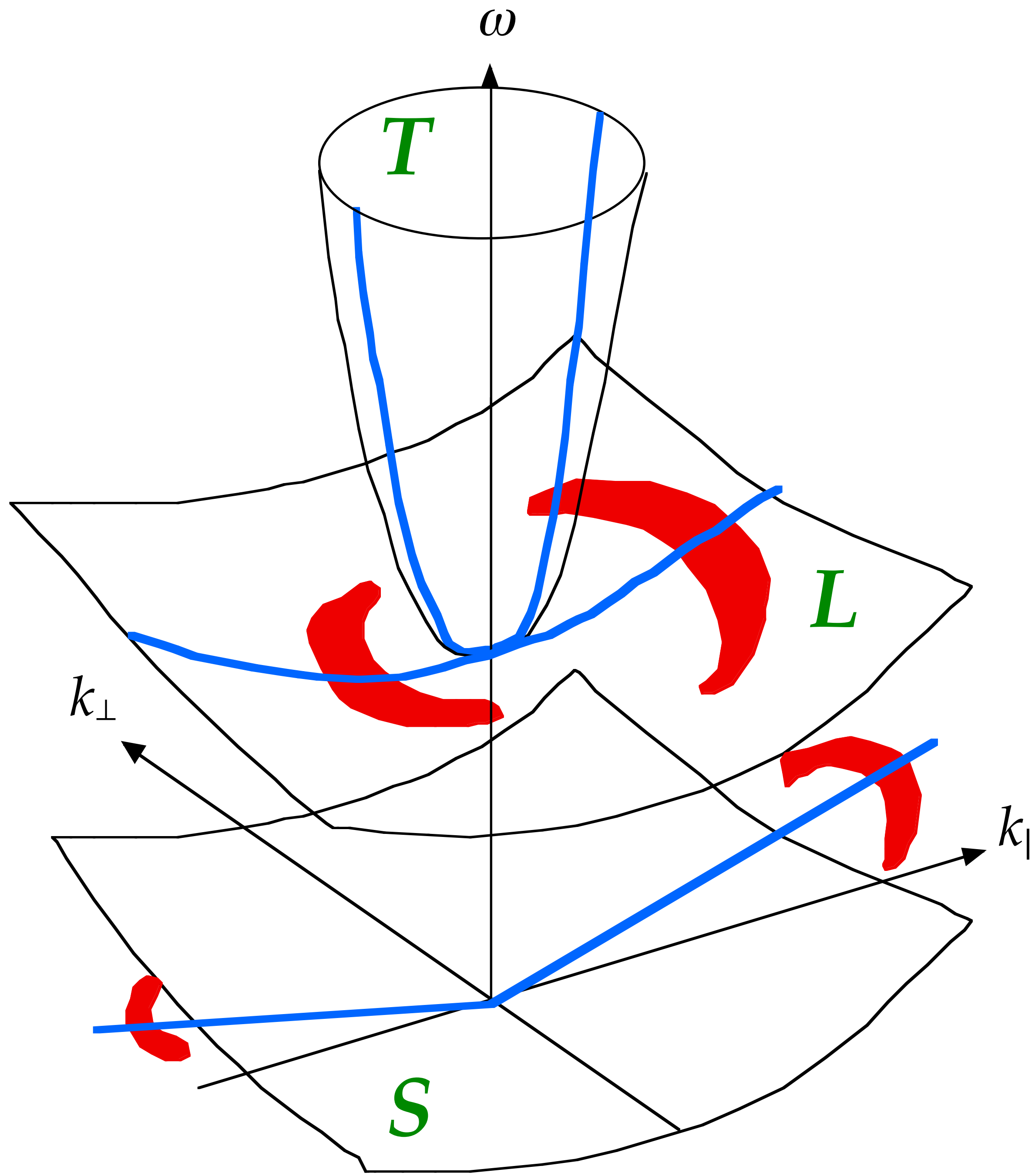

ω
T
L
k⊥
k‖
S

Backscattered L wave



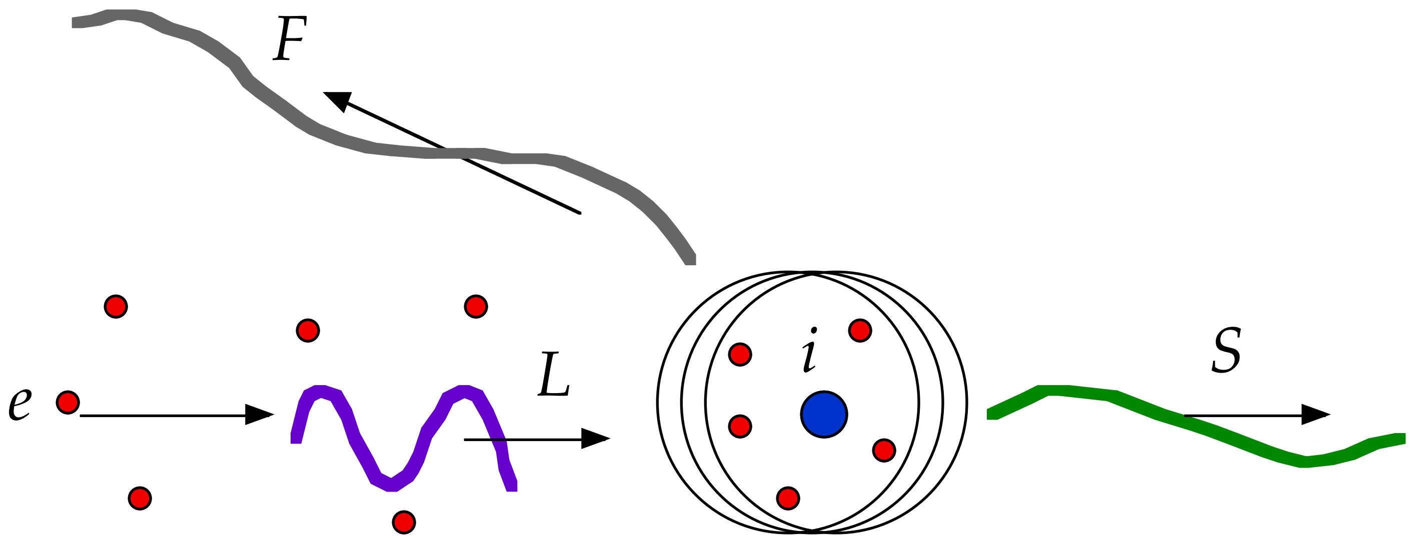


ω
T
2ω_pe
V_0
L′
L
ω_pe
S
k



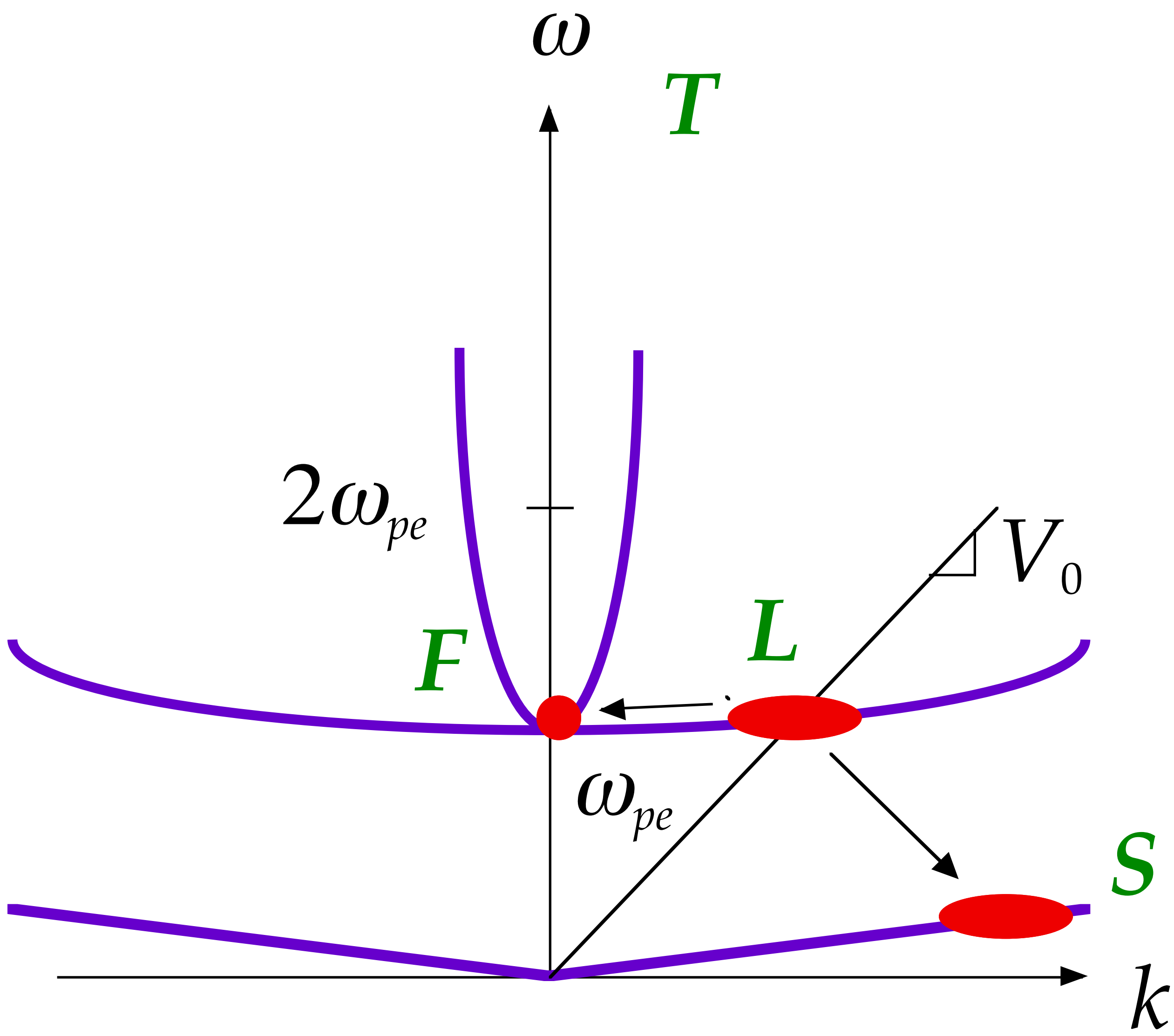
ω
T
L
$k_\perp$
$k_\parallel$
S

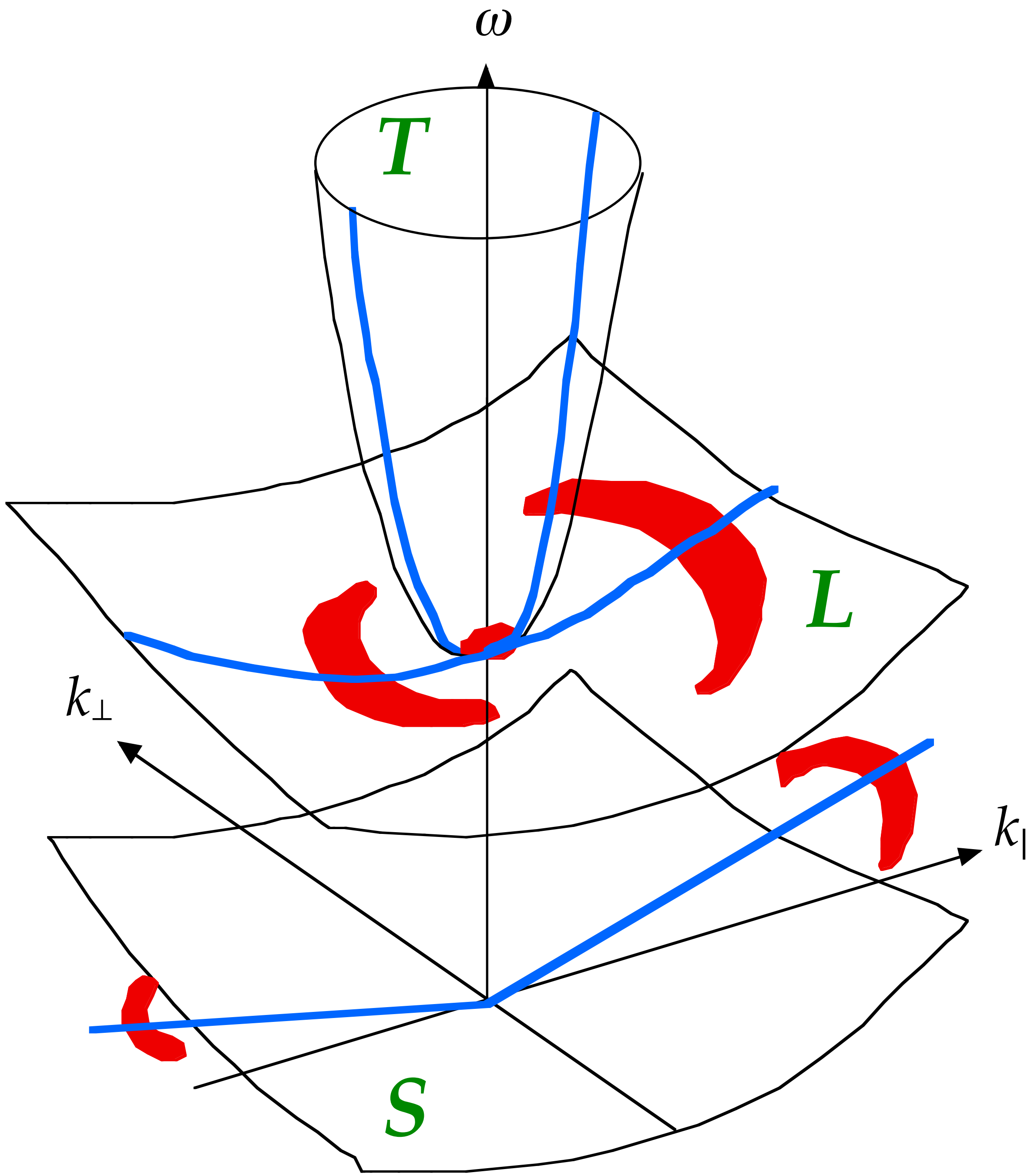
F Emission



Decay of L mode into S and T at Fundamental

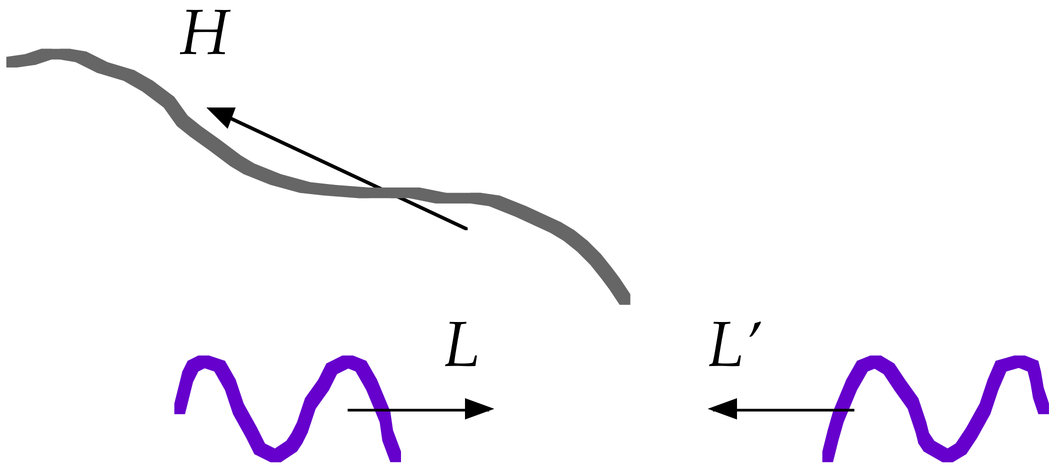
F Emission





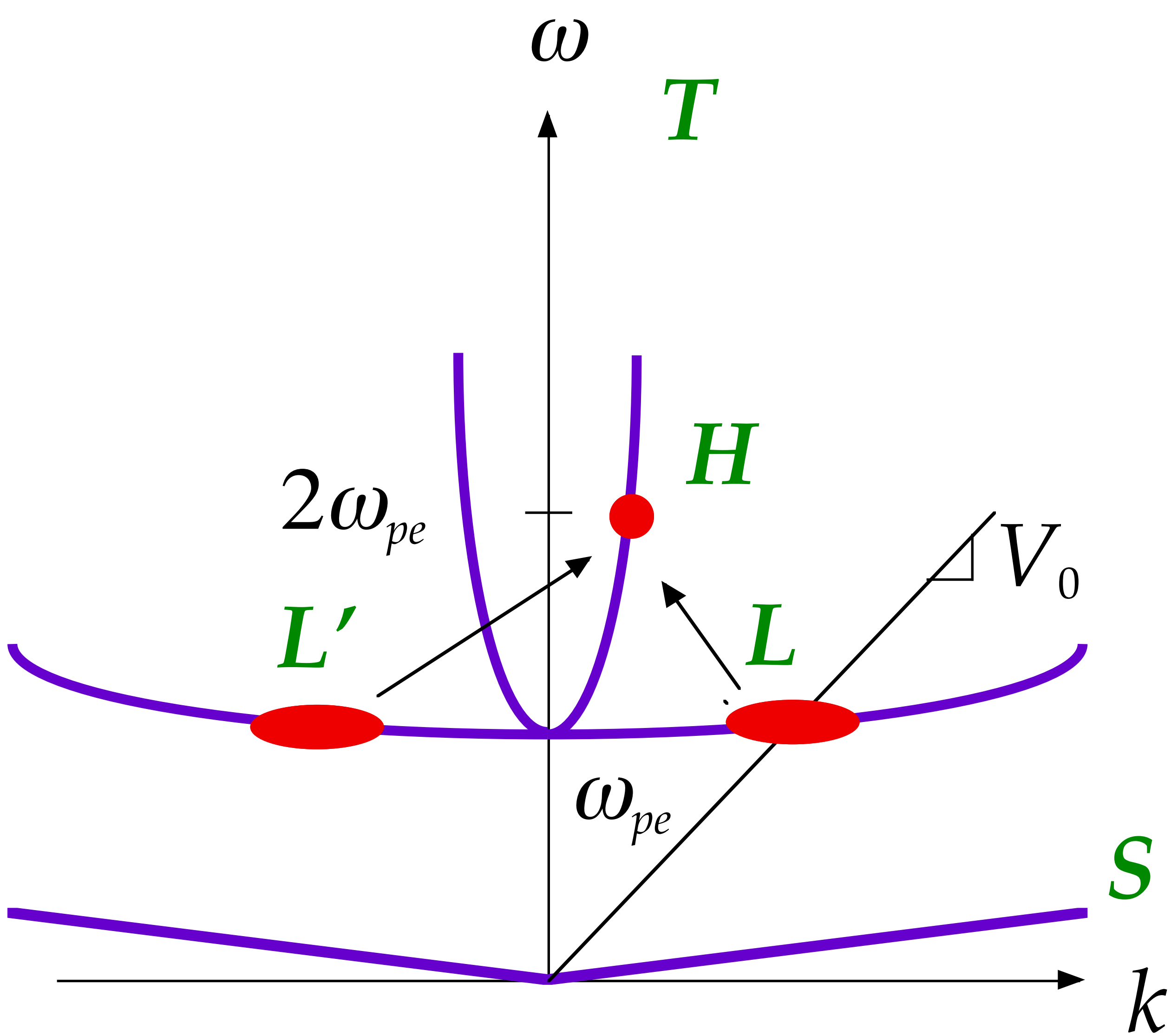
ω
T
L
k⊥
k||
S

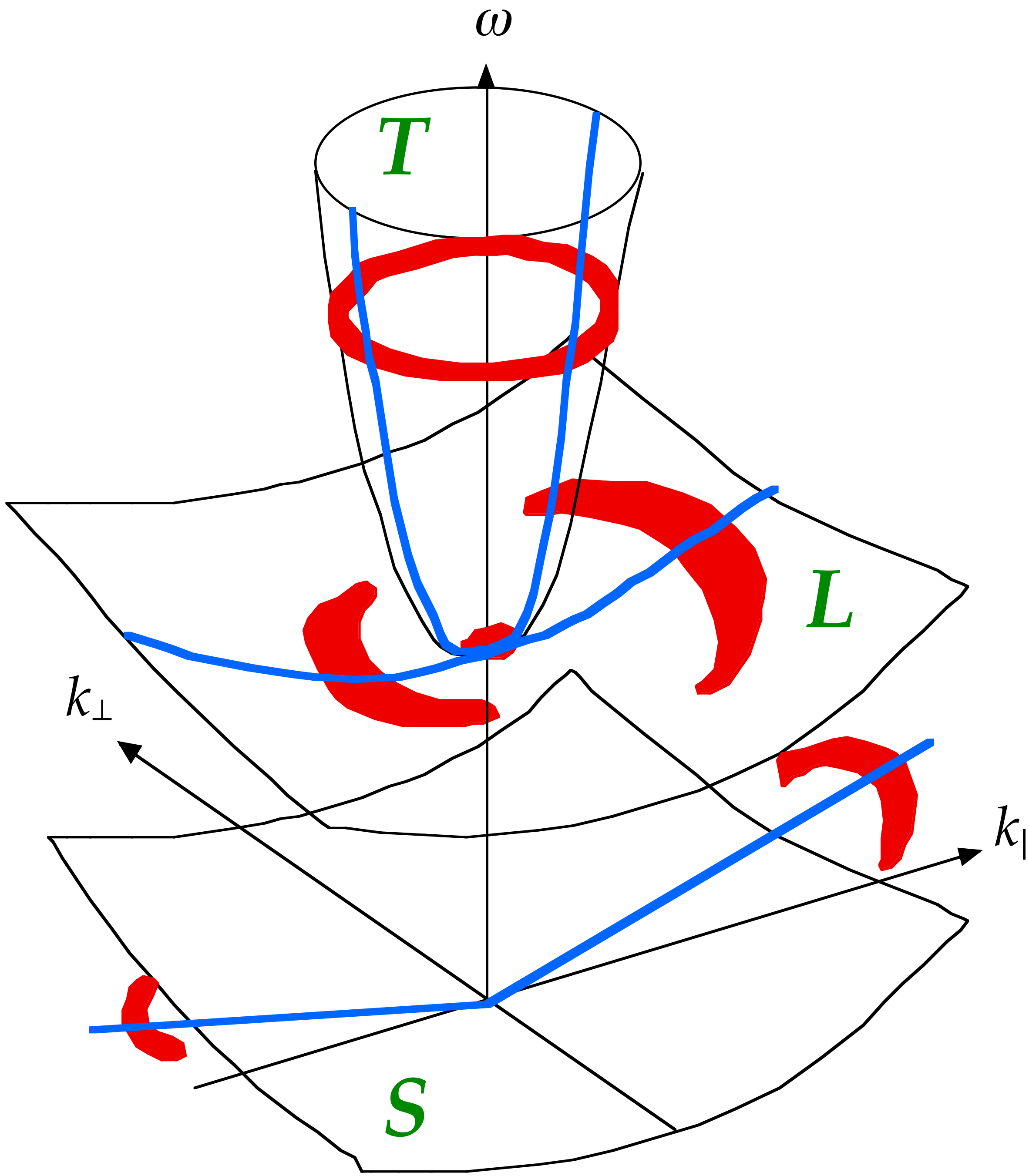
H Emission



Merging of Primary L and Backscattered L’ modes into T mode at Harmonic

H Emission




ω
T
L
k⊥
k||
S

Plasma emission scenario

- Many advances have been made over six decades of research.
- **We have recently carried out EM weak turbulence calculation of plasma emission from beam-plasma instability to radiation generation [Ziebell, Yoon, Gaelzer, Pavan, *ApJL*, 2014; *ApJ*, 2015]**

Langmuir wave kinetic equation

$$\frac{\partial I_{\mathbf{k}}^{\sigma L}}{\partial t} = \frac{\pi\omega_{pe}^2}{k^2}\int d\mathbf{v}\,\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v})\left(\frac{ne^2}{\pi}f_e + \sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}}^{\sigma L}\mathbf{k}\cdot\frac{\partial f_e}{\partial \mathbf{v}}\right)$$

Spontaneous & induced emission

$$+2\sum_{\sigma',\sigma''=\pm1}\sigma\omega_{\mathbf{k}}^L\int d\mathbf{k}' V_{\mathbf{k},\mathbf{k}'}^L\,\delta(\sigma\omega_{\mathbf{k}}^L - \sigma'\omega_{\mathbf{k}'}^L - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^S)$$

$$\times\left(\sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} - \sigma'\omega_{\mathbf{k}'}^L I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} I_{\mathbf{k}}^{\sigma L} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma L}\right)$$

(*L* -> *L*+*S*) three wave decay

$$-\frac{\pi e^2}{m_e^2\omega_{pe}^2}\sigma\omega_{\mathbf{k}}^L\sum_{\sigma'=\pm1}\int d\mathbf{k}'\int d\mathbf{v}\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2}\delta[\sigma\omega_{\mathbf{k}}^L - \sigma'\omega_{\mathbf{k}'}^L - (\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}]$$

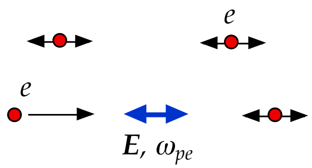
$$\times\left(\frac{ne^2}{\pi\omega_{pe}^2}(\sigma'\omega_{\mathbf{k}'}^L I_{\mathbf{k}}^{\sigma L} - \sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}'}^{\sigma' L})(f_e + f_i)\right.$$

$$\left.+(\sigma'\omega_{\mathbf{k}'}^L - \sigma\omega_{\mathbf{k}}^L)I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma L}(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial f_e}{\partial\mathbf{v}} - \frac{m_e}{m_i}I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma L}(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial f_i}{\partial\mathbf{v}}\right)$$

(*L* -> *L*+*e*) spontaneous & induced scattering

Langmuir wave kinetic equation

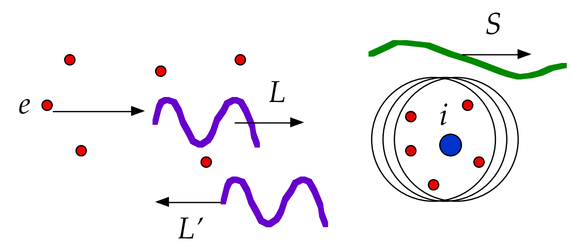
$$\frac{\partial I_{\mathbf{k}}^{\sigma L}}{\partial t} = \frac{\pi\omega_{pe}^2}{k^2}\int d\mathbf{v}\,\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v})\left(\frac{ne^2}{\pi}f_e + \sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}}^{\sigma L}\mathbf{k}\cdot\frac{\partial f_e}{\partial\mathbf{v}}\right)$$



Spontaneous & induced emission

$$+2\sum_{\sigma',\sigma''=\pm1}\sigma\omega_{\mathbf{k}}^L\int d\mathbf{k}' V_{\mathbf{k},\mathbf{k}'}^L\,\delta(\sigma\omega_{\mathbf{k}}^L - \sigma'\omega_{\mathbf{k}'}^L - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^S)$$

$$\times\left(\sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} - \sigma'\omega_{\mathbf{k}'}^L I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} I_{\mathbf{k}}^{\sigma L} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma L}\right)$$



(*L* -> *L*+*S*) three wave decay

$$-\frac{\pi e^2}{m_e^2\omega_{pe}^2}\sigma\omega_{\mathbf{k}}^L\sum_{\sigma'=\pm1}\int d\mathbf{k}'\int d\mathbf{v}\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2}\delta[\sigma\omega_{\mathbf{k}}^L - \sigma'\omega_{\mathbf{k}'}^L - (\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}]$$

$$\times\left(\frac{ne^2}{\pi\omega_{pe}^2}(\sigma'\omega_{\mathbf{k}'}^L I_{\mathbf{k}}^{\sigma L} - \sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}'}^{\sigma' L})(f_e + f_i)\right.$$

$$\left.+(\sigma'\omega_{\mathbf{k}'}^L - \sigma\omega_{\mathbf{k}}^L)I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma L}(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial f_e}{\partial\mathbf{v}} - \frac{m_e}{m_i}I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma L}(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial f_i}{\partial\mathbf{v}}\right)$$

(*L* -> *L*+*e*) spontaneous & induced scattering

Ion-acoustic wave kinetic equation

$$\frac{\partial}{\partial t}\frac{I_{\mathbf{k}}^{\sigma S}}{\mu_{\mathbf{k}}} = \frac{\pi\mu_{\mathbf{k}}\omega_{pe}^2}{k^2}\int d\mathbf{v}\delta(\sigma\omega_{\mathbf{k}}^S - \mathbf{k}\cdot\mathbf{v})\left[\frac{ne^2}{\pi}[f_e + f_i]\right.$$

$$\left. + \pi\sigma\omega_{\mathbf{k}}^L\left(\mathbf{k}\cdot\frac{\partial f_e}{\partial \mathbf{v}}_e + \frac{m_e}{m_i}\mathbf{k}\cdot\frac{\partial f_i}{\partial \mathbf{v}}\right)\frac{I_{\mathbf{k}}^{\sigma S}}{\mu_{\mathbf{k}}}\right]$$

Spontaneous & induced emission

$$+\sum_{\sigma',\sigma''=\pm 1}\sigma\omega_{\mathbf{k}}^L\int d\mathbf{k}' V_{\mathbf{k},\mathbf{k}'}^S\delta(\sigma\omega_{\mathbf{k}}^S - \sigma'\omega_{\mathbf{k}'}^L - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L)$$

$$\times\left(\sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L} - \frac{\sigma'\omega_{\mathbf{k}'}^L I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L} I_{\mathbf{k}}^{\sigma S}}{\mu_{\mathbf{k}}} - \frac{\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma S}}{\mu_{\mathbf{k}}}\right)$$

(*S* -> *L*+*L*) three wave decay

Particle kinetic equation

$$\frac{\partial F_a}{\partial t} = \frac{\pi e^2}{m_e^2}\sum_{\sigma=\pm 1}\sum_{\alpha=L,S}\int d\mathbf{k}\frac{\mathbf{k}}{k}\bullet\frac{\partial}{\partial \mathbf{v}}\delta(\sigma\omega_{\mathbf{k}}^\alpha - \mathbf{k}\bullet\mathbf{v})\left(\frac{m_e\sigma\omega_{\mathbf{k}}^\alpha}{4\pi^2 k}F_a + I_{\mathbf{k}}^{\sigma\alpha}\frac{\mathbf{k}}{k}\bullet\frac{\partial F_a}{\partial \mathbf{v}}\right)$$

Transverse EM wave kinetic equation

$$\frac{\partial}{\partial t}\frac{I_{\mathbf{k}}^{\sigma T}}{2} = \sum_{\sigma',\sigma''=\pm1} \int d\mathbf{k}' V_{\mathbf{k},\mathbf{k}'}^{TLL} \delta(\sigma\omega_{\mathbf{k}}^{T} - \sigma'\omega_{\mathbf{k}'}^{L} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L})$$

$$\times\left(\sigma\omega_{\mathbf{k}}^{T} I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L} - \frac{\sigma'\omega_{\mathbf{k}'}^{L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L} I_{\mathbf{k}}^{\sigma T}}{2} - \frac{\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L} I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma T}}{2}\right)$$

(L+L -> T) Harmonic emission

$$+ \sum_{\sigma',\sigma''=\pm1} \int d\mathbf{k}' V_{\mathbf{k},\mathbf{k}'}^{TLS} \delta(\sigma\omega_{\mathbf{k}}^{T} - \sigma'\omega_{\mathbf{k}'}^{L} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{S})\left(\frac{\sigma\omega_{\mathbf{k}}^{T} I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S}}{\mu_{\mathbf{k}-\mathbf{k}'}} - \frac{\sigma'\omega_{\mathbf{k}'}^{L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} I_{\mathbf{k}}^{\sigma T}}{2\mu_{\mathbf{k}-\mathbf{k}'}} - \frac{\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L} I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma T}}{2}\right)$$

(L+S -> T) Fundamental emission by decay

$$+ \sum_{\sigma',\sigma''=\pm1} \int d\mathbf{k}' V_{\mathbf{k},\mathbf{k}'}^{TTL} \delta(\sigma\omega_{\mathbf{k}}^{T} - \sigma'\omega_{\mathbf{k}'}^{T} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L})\left(\frac{\sigma\omega_{\mathbf{k}}^{T} I_{\mathbf{k}'}^{\sigma' T} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' T}}{2} - \frac{\sigma'\omega_{\mathbf{k}'}^{T} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L} I_{\mathbf{k}}^{\sigma T}}{2} - \frac{\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L} I_{\mathbf{k}'}^{\sigma' T} I_{\mathbf{k}}^{\sigma T}}{4}\right)$$

(L+T -> T) Higher-harmonic emission

$$+ \sum_{\sigma'=\pm1} \int d\mathbf{k}' \int d\mathbf{v}\, U_{\mathbf{k},\mathbf{k}'}^{T} \delta[\sigma\omega_{\mathbf{k}}^{T} - \sigma'\omega_{\mathbf{k}'}^{L} - (\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}]$$

(L+I -> T) Fundamental emission by scattering

$$\times\left[\frac{ne^2}{\pi\omega_{pe}^2}\left(\sigma\omega_{\mathbf{k}}^{T} I_{\mathbf{k}'}^{\sigma' L} - \frac{\sigma'\omega_{\mathbf{k}'}^{T} I_{\mathbf{k}}^{\sigma T}}{2}\right)(f_e + f_i) + \pi\frac{m_e}{m_i} I_{\mathbf{k}'}^{\sigma' L}\frac{I_{\mathbf{k}}^{\sigma L}}{2}(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial f_i}{\partial \mathbf{v}}\right]$$

Transverse EM wave kinetic equation

$$\frac{\partial}{\partial t}\frac{I_{\mathbf{k}}^{\sigma T}}{2} = \sum_{\sigma',\sigma''=\pm 1}\int d\mathbf{k}' V_{\mathbf{k},\mathbf{k}'}^{TLL}\,\delta(\sigma\omega_{\mathbf{k}}^{T} - \sigma'\omega_{\mathbf{k}'}^{L} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L})$$

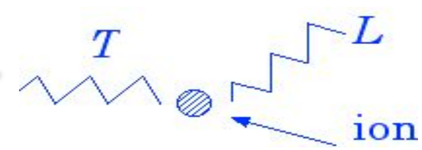
$$\times\left(\sigma\omega_{\mathbf{k}}^{T} I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L} - \frac{\sigma'\omega_{\mathbf{k}'}^{L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L} I_{\mathbf{k}}^{\sigma T}}{2} - \frac{\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L} I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma T}}{2}\right)$$

$$+\sum_{\sigma',\sigma''=\pm 1}\int d\mathbf{k}' V_{\mathbf{k},\mathbf{k}'}^{TLS}\,\delta(\sigma\omega_{\mathbf{k}}^{T} - \sigma'\omega_{\mathbf{k}'}^{L} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{S})\left(\frac{\sigma\omega_{\mathbf{k}}^{T} I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S}}{\mu_{\mathbf{k}-\mathbf{k}'}} - \frac{\sigma'\omega_{\mathbf{k}'}^{L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} I_{\mathbf{k}}^{\sigma T}}{2\mu_{\mathbf{k}-\mathbf{k}'}} - \frac{\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L} I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma T}}{2}\right)$$

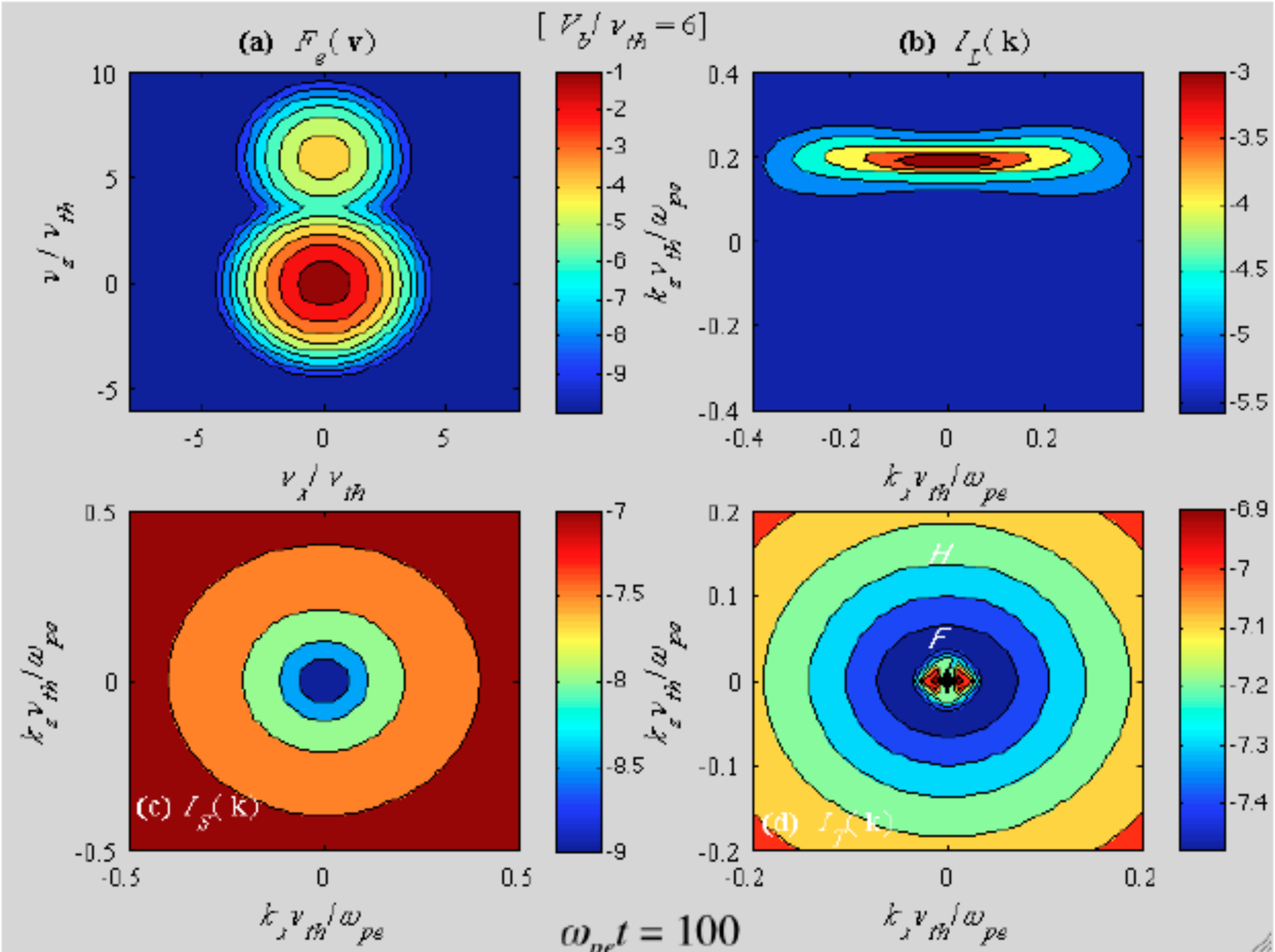
$$+\sum_{\sigma',\sigma''=\pm 1}\int d\mathbf{k}' V_{\mathbf{k},\mathbf{k}'}^{TTL}\,\delta(\sigma\omega_{\mathbf{k}}^{T} - \sigma'\omega_{\mathbf{k}'}^{T} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L})\left(\frac{\sigma\omega_{\mathbf{k}}^{T} I_{\mathbf{k}'}^{\sigma' T} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' T}}{2} - \frac{\sigma'\omega_{\mathbf{k}'}^{T} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L} I_{\mathbf{k}}^{\sigma T}}{2} - \frac{\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L} I_{\mathbf{k}'}^{\sigma' T} I_{\mathbf{k}}^{\sigma T}}{4}\right)$$

(*L*+*T* -> *T*) Higher-harmonic emission

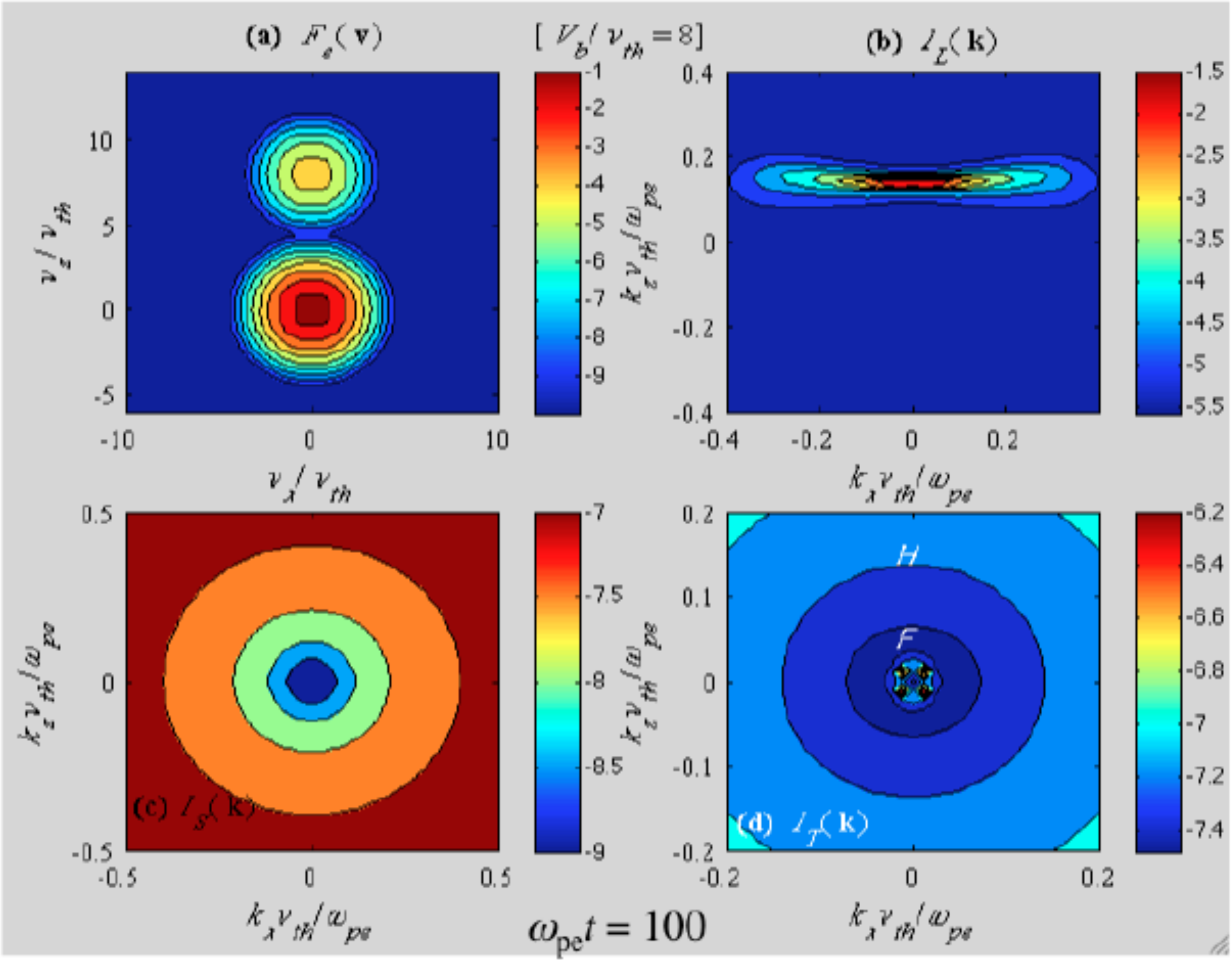
$$+\sum_{\sigma'=\pm 1}\int d\mathbf{k}'\int d\mathbf{v}\, U_{\mathbf{k},\mathbf{k}'}^{T}\,\delta[\sigma\omega_{\mathbf{k}}^{T} - \sigma'\omega_{\mathbf{k}'}^{L} - (\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}]$$



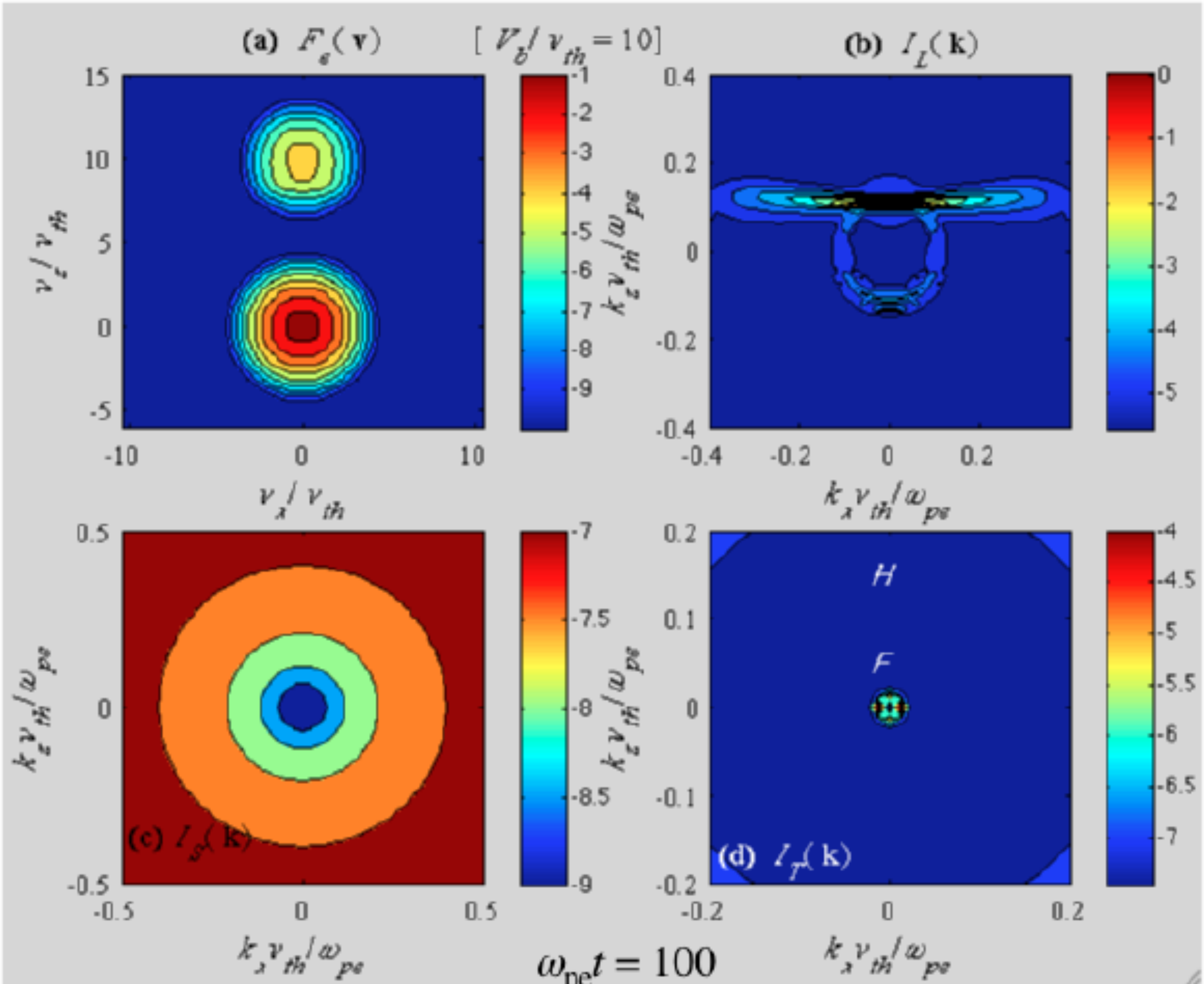
$$\times\left[\frac{ne^2}{\pi\omega_{pe}^2}\left(\sigma\omega_{\mathbf{k}}^{T} I_{\mathbf{k}'}^{\sigma' L} - \frac{\sigma'\omega_{\mathbf{k}'}^{T} I_{\mathbf{k}}^{\sigma T}}{2}\right)(f_e + f_i) + \pi\frac{m_e}{m_i} I_{\mathbf{k}'}^{\sigma' L}\frac{I_{\mathbf{k}}^{\sigma L}}{2}(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial f_i}{\partial \mathbf{v}}\right]$$



[$V_b/v_{th} = 6$]
(a) $F_e(\mathbf{v})$
(b) $I_L(\mathrm{k})$
(c) $I_S(\mathrm{k})$
(d) $I_T(\mathrm{k})$
v_z/v_{th}
v_x/v_{th}
$k_z v_{th}/\omega_{pe}$
$k_x v_{th}/\omega_{pe}$
H
F
$\omega_{pe} t = 100$

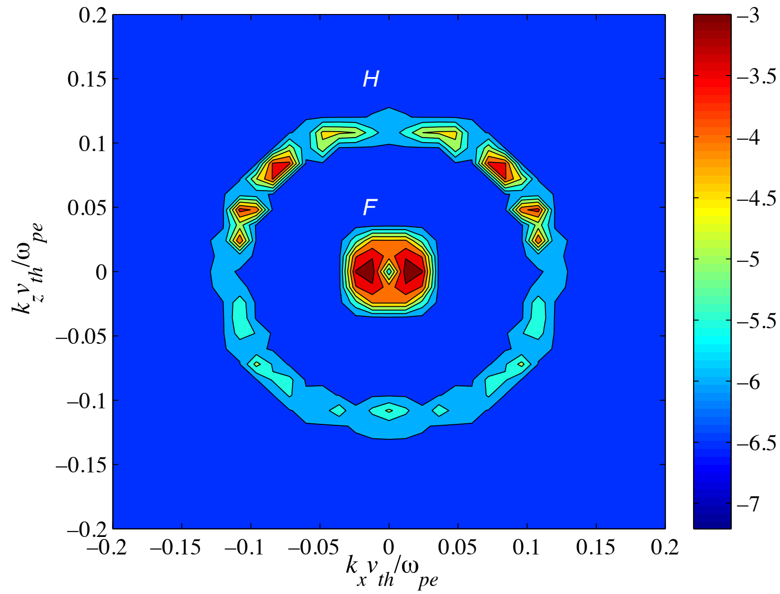


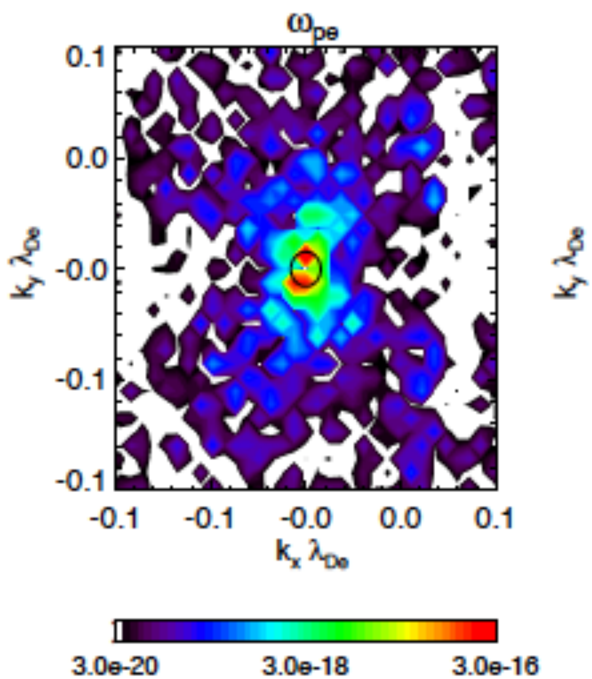
(a) $F_e(\mathbf{v})$
$[V_b/v_{th} = 8]$
(b) $I_L(\mathbf{k})$
v_z/v_{th}
v_x/v_{th}
$k_z v_{th}/\omega_{pe}$
$k_x v_{th}/\omega_{pe}$
(c) $I_S(\mathbf{k})$
(d) $I_T(\mathbf{k})$
H
F
$\omega_{pe}t = 100$



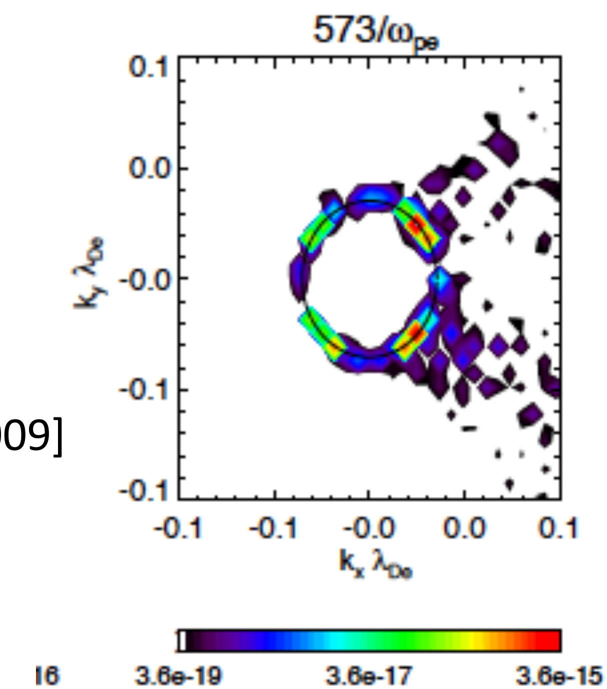
(a) $F_e(\mathbf{v})$
$[V_b/v_{th} = 10]$
(b) $I_L(\mathbf{k})$
v_z/v_{th}
v_x/v_{th}
$k_z v_{th}/\omega_{pe}$
$k_x v_{th}/\omega_{pe}$
(c) $I_S(\mathbf{k})$
(d) $I_T(\mathbf{k})$
H
F
$\omega_{pe}t = 100$

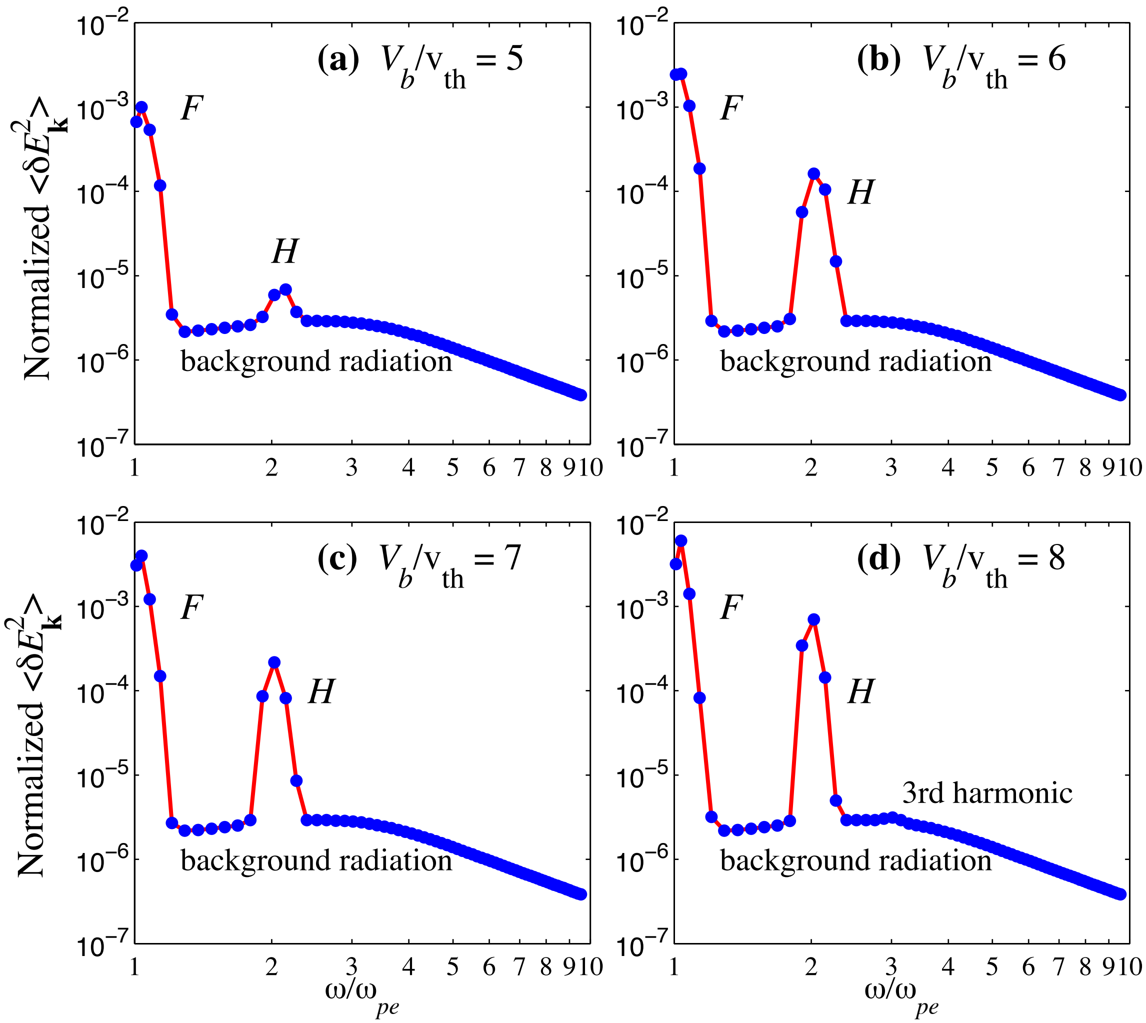
EM Weak turb simulation
[*Ziebell et al.*, 2015]



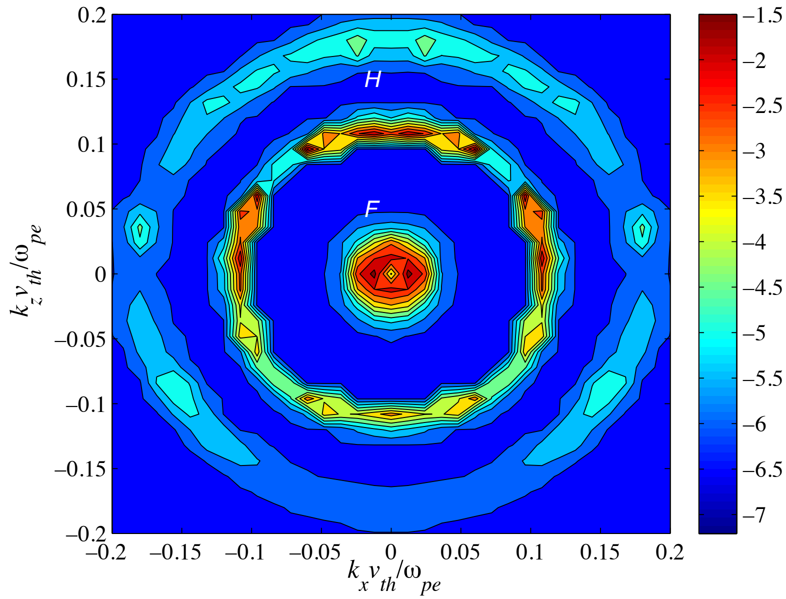


PIC simulation
[*Rhee et al.*, 2009]

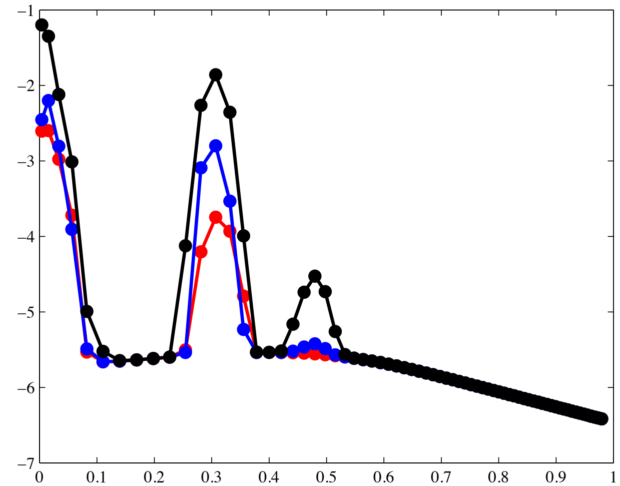




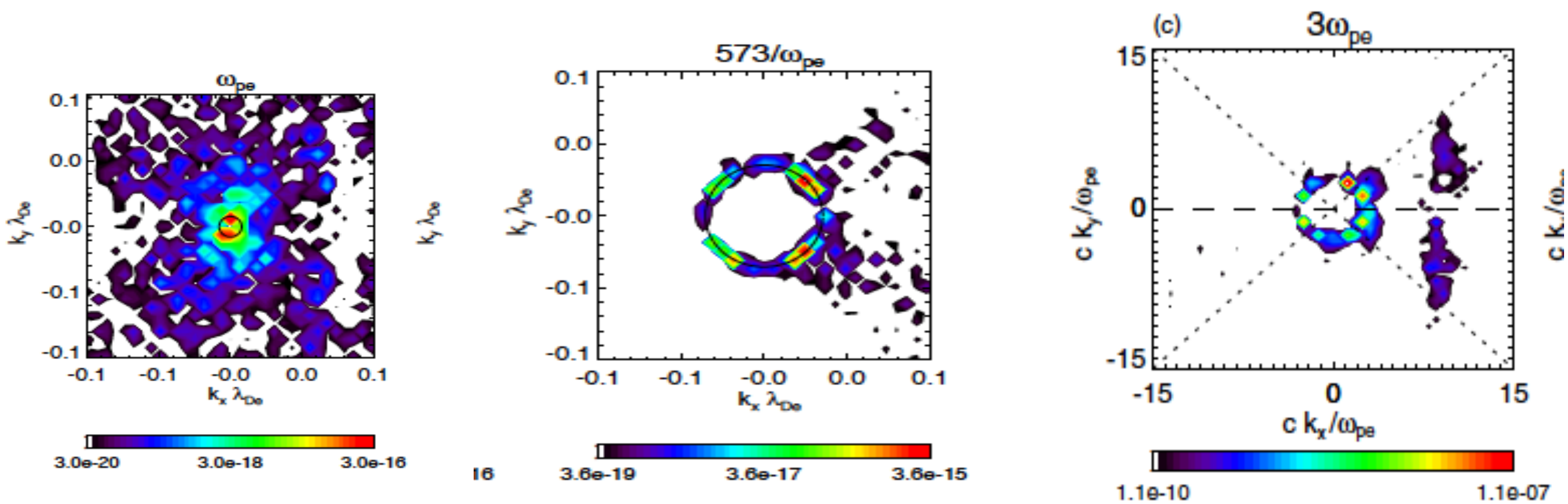
EM Weak turb simulation
[*Ziebell et al.,* 2015]



EM Weak turb simulation
[*Ziebell et al.,* 2015]



PIC simulation [*Rhee et al.,* 2009]



Transverse EM wave kinetic equation

$$\frac{\partial}{\partial t}\frac{I_{\mathbf{k}}^{\sigma T}}{2} = \sum_{\sigma',\sigma''=\pm 1}\int d\mathbf{k}' V_{\mathbf{k},\mathbf{k}'}^{TLL}\delta(\sigma\omega_{\mathbf{k}}^{T} - \sigma'\omega_{\mathbf{k}'}^{L} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L})$$

$$\times\left(\sigma\omega_{\mathbf{k}}^{T} I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L} - \frac{\sigma'\omega_{\mathbf{k}'}^{L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L} I_{\mathbf{k}}^{\sigma T}}{2} - \frac{\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L} I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma T}}{2}\right)$$

(*L*+*L* -> *T*) Harmonic emission

$$+\sum_{\sigma',\sigma''=\pm 1}\int d\mathbf{k}' V_{\mathbf{k},\mathbf{k}'}^{TLS}\delta(\sigma\omega_{\mathbf{k}}^{T} - \sigma'\omega_{\mathbf{k}'}^{L} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{S})\left(\frac{\sigma\omega_{\mathbf{k}}^{T} I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S}}{\mu_{\mathbf{k}-\mathbf{k}'}} - \frac{\sigma'\omega_{\mathbf{k}'}^{L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} I_{\mathbf{k}}^{\sigma T}}{2\mu_{\mathbf{k}-\mathbf{k}'}} - \frac{\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L} I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma T}}{2}\right)$$

(*L*+*S* -> *T*) Fundamental emission by decay

$$+\sum_{\sigma',\sigma''=\pm 1}\int d\mathbf{k}' V_{\mathbf{k},\mathbf{k}'}^{TTL}\delta(\sigma\omega_{\mathbf{k}}^{T} - \sigma'\omega_{\mathbf{k}'}^{T} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L})\left(\frac{\sigma\omega_{\mathbf{k}}^{T} I_{\mathbf{k}'}^{\sigma' T} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' T}}{2} - \frac{\sigma'\omega_{\mathbf{k}'}^{T} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L} I_{\mathbf{k}}^{\sigma T}}{2} - \frac{\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L} I_{\mathbf{k}'}^{\sigma' T} I_{\mathbf{k}}^{\sigma T}}{4}\right)$$

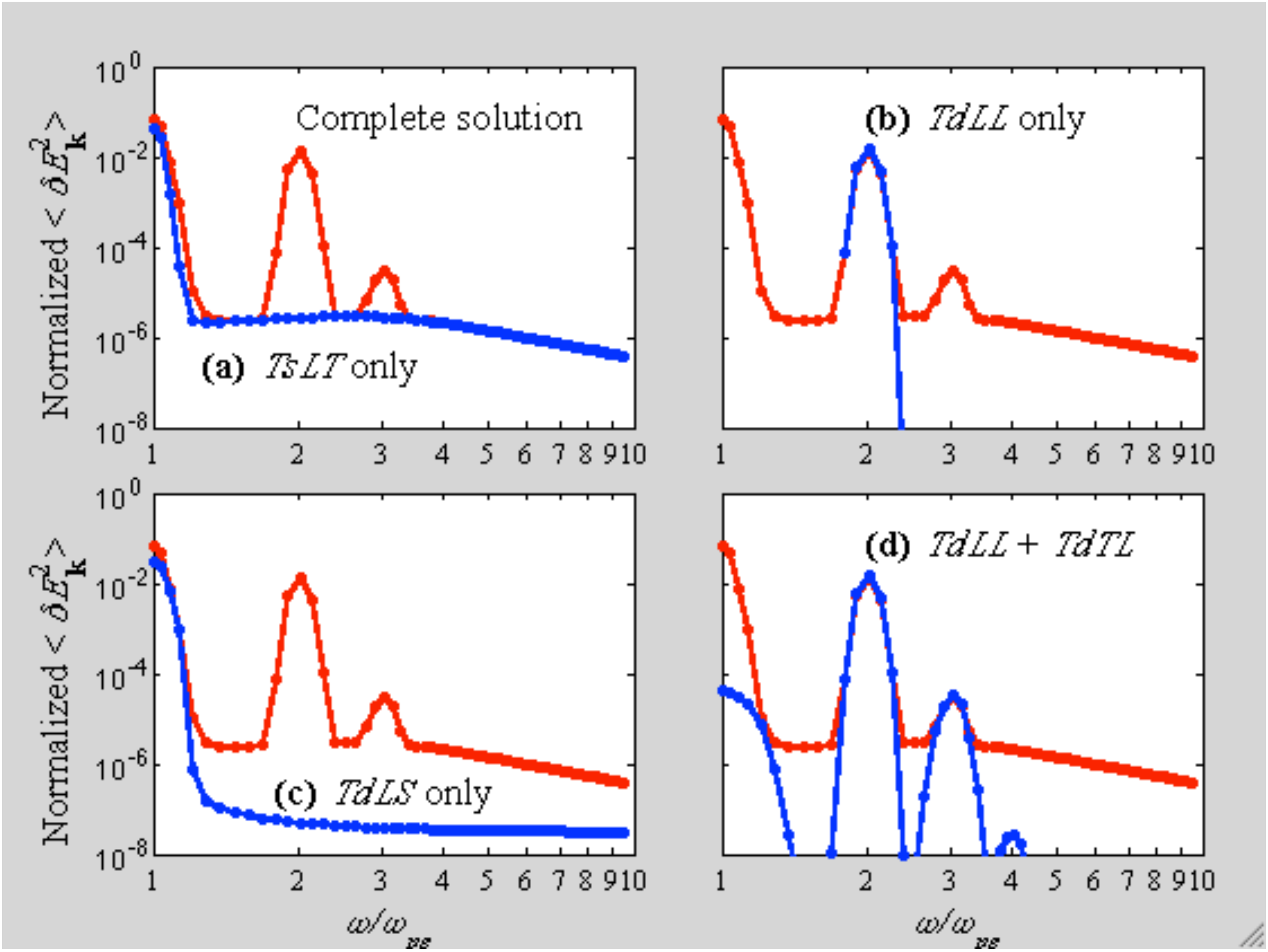
(*L*+*T* -> *T*) Higher-harmonic emission

$$+\sum_{\sigma'=\pm 1}\int d\mathbf{k}'\int d\mathbf{v} U_{\mathbf{k},\mathbf{k}'}^{T}\delta[\sigma\omega_{\mathbf{k}}^{T} - \sigma'\omega_{\mathbf{k}'}^{L} - (\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}]$$

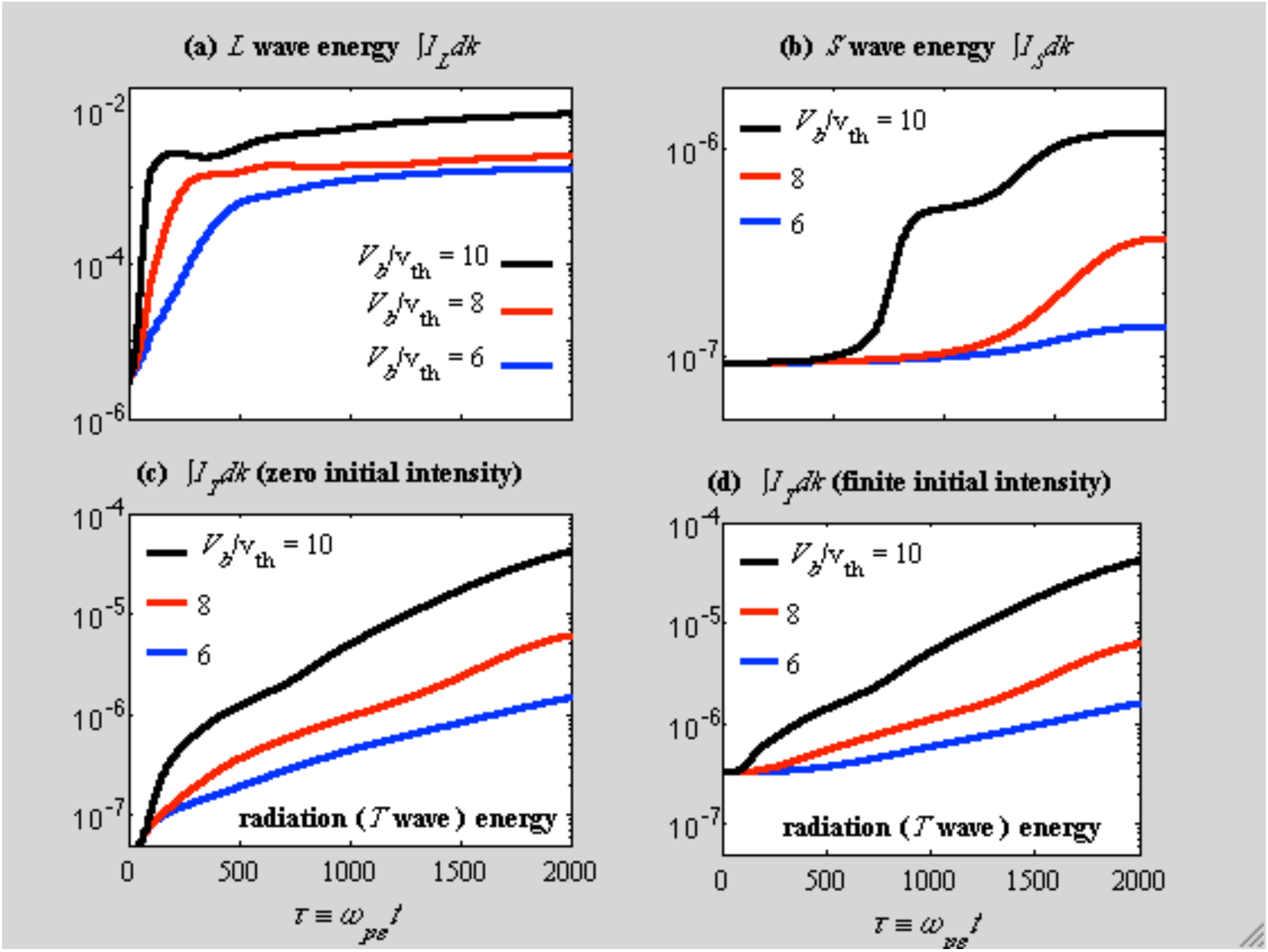
(*L*+*I* -> *T*) Fundamental emission by scattering

$$\times\left[\frac{ne^2}{\pi\omega_{pe}^2}\left(\sigma\omega_{\mathbf{k}}^{T} I_{\mathbf{k}'}^{\sigma' L} - \frac{\sigma'\omega_{\mathbf{k}'}^{T} I_{\mathbf{k}}^{\sigma T}}{2}\right)(f_e + f_i) + \pi\frac{m_e}{m_i} I_{\mathbf{k}'}^{\sigma' L}\frac{I_{\mathbf{k}}^{\sigma L}}{2}(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial f_i}{\partial\mathbf{v}}\right]$$

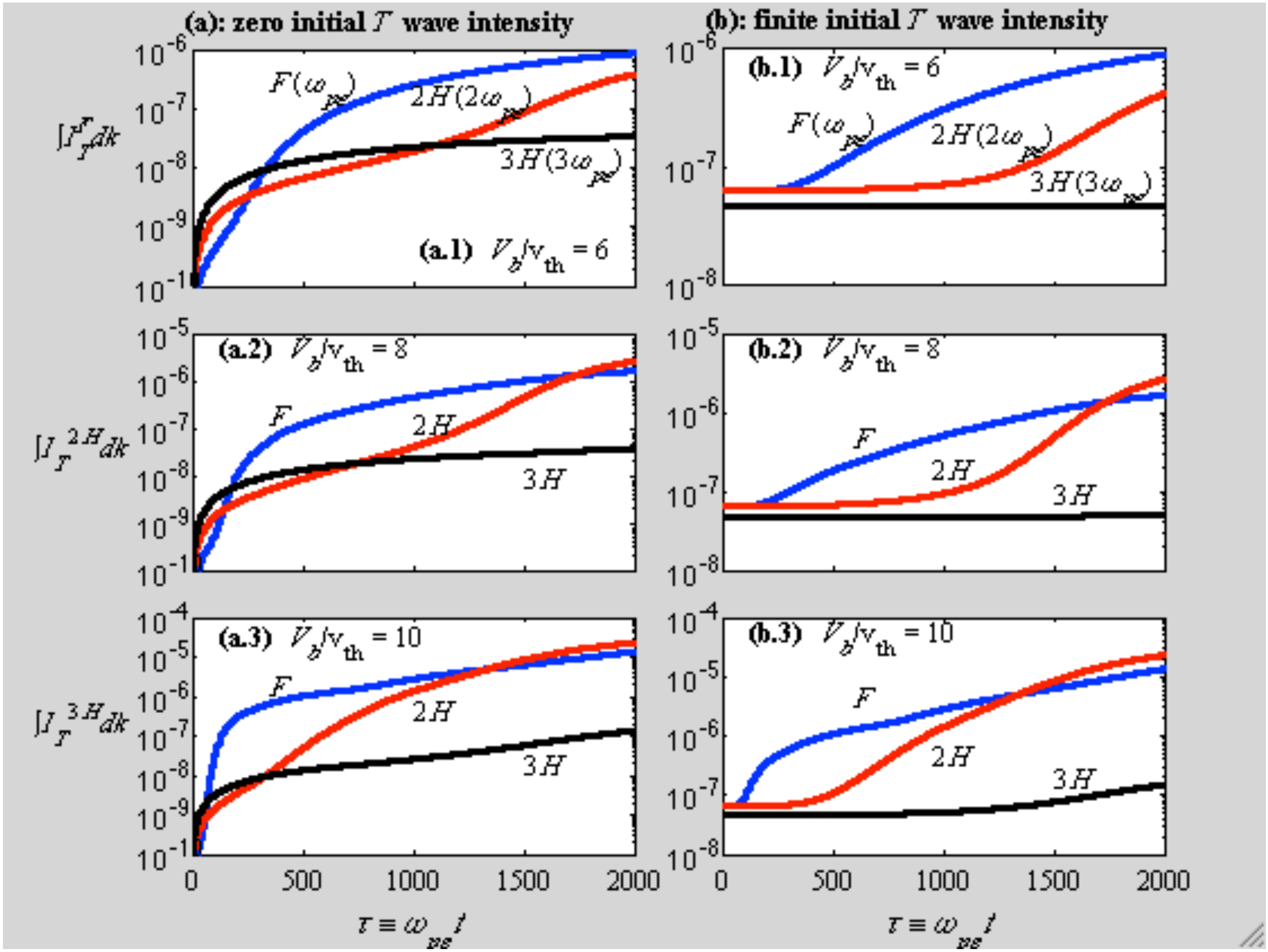
EM Weak turb simulation
[*Ziebell et al.,* 2015]

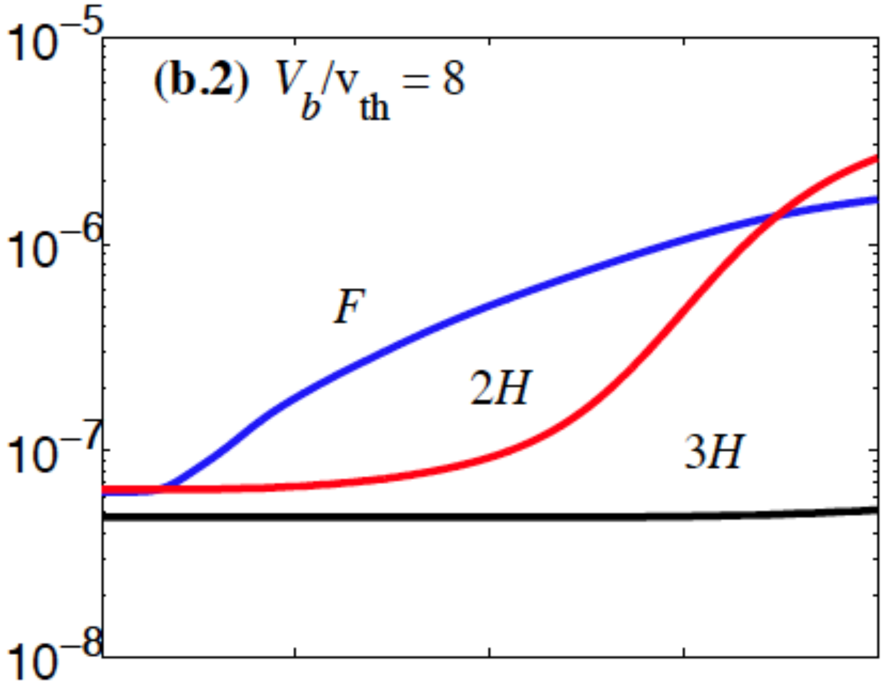


EM Weak turb simulation
[*Ziebell et al.,* 2015]

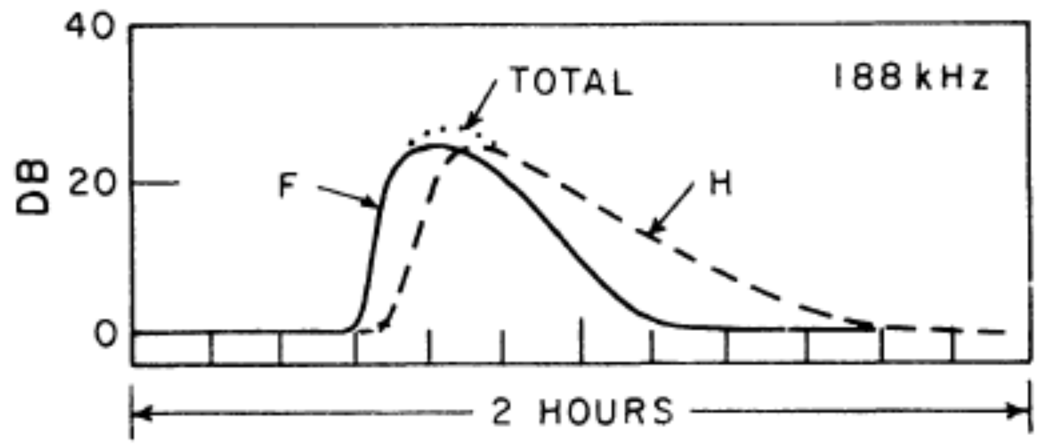


EM Weak turb simulation
[*Ziebell et al.,* 2015]



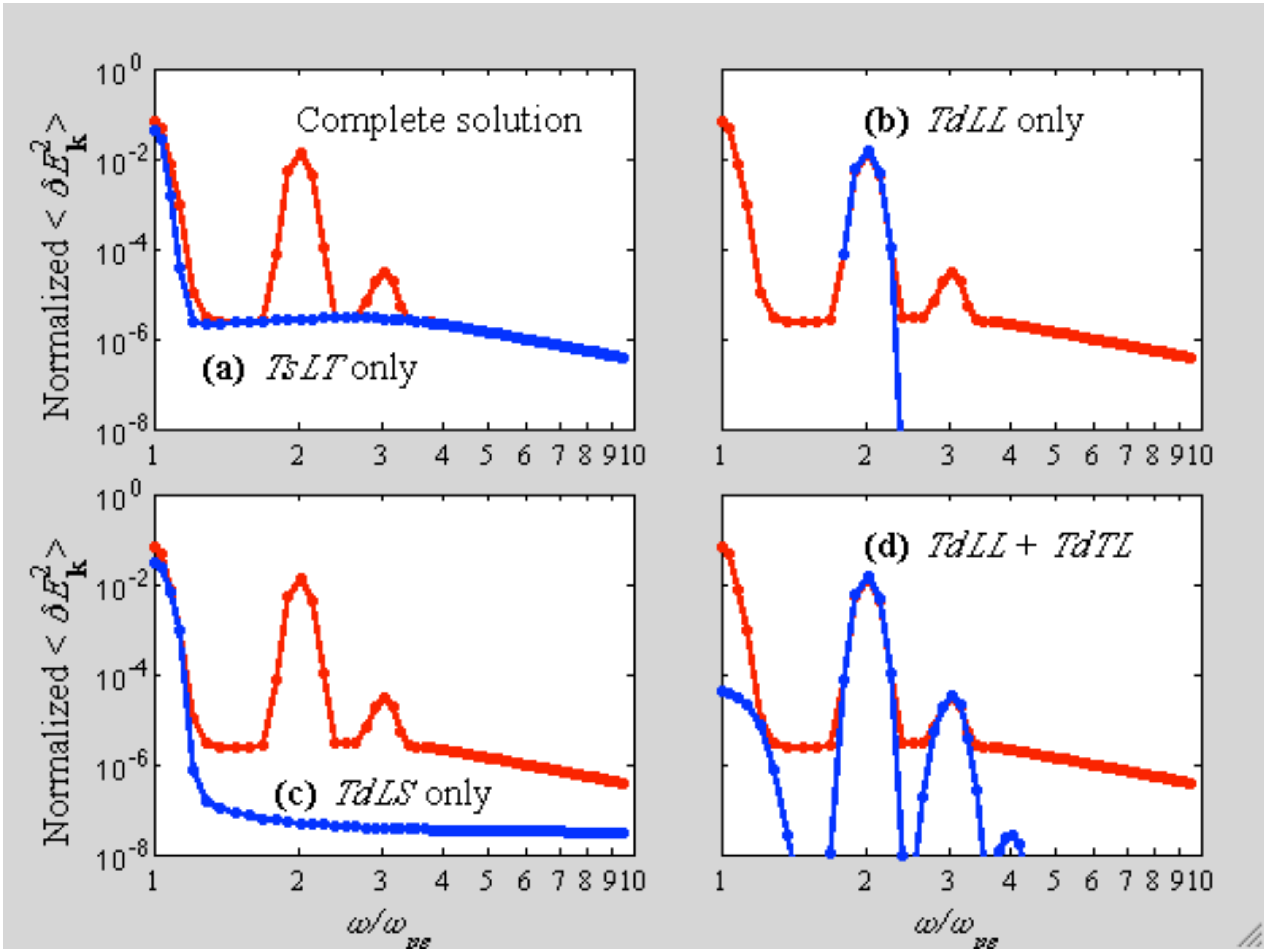


EM Weak turb simulation
[*Ziebell et al.,* 2015]



Dulk et al. 1984

Background Radiation

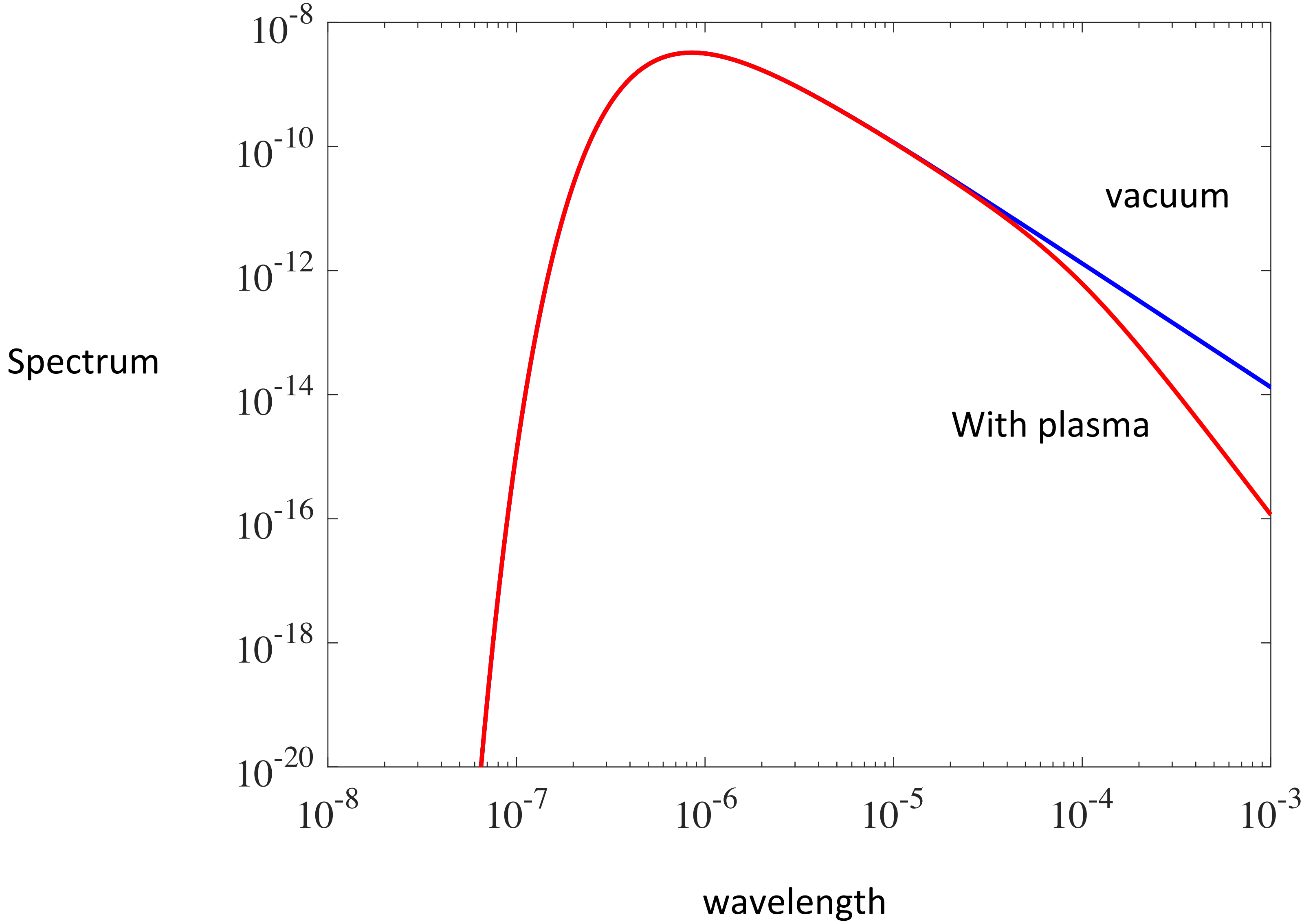


Blackbody Radiation

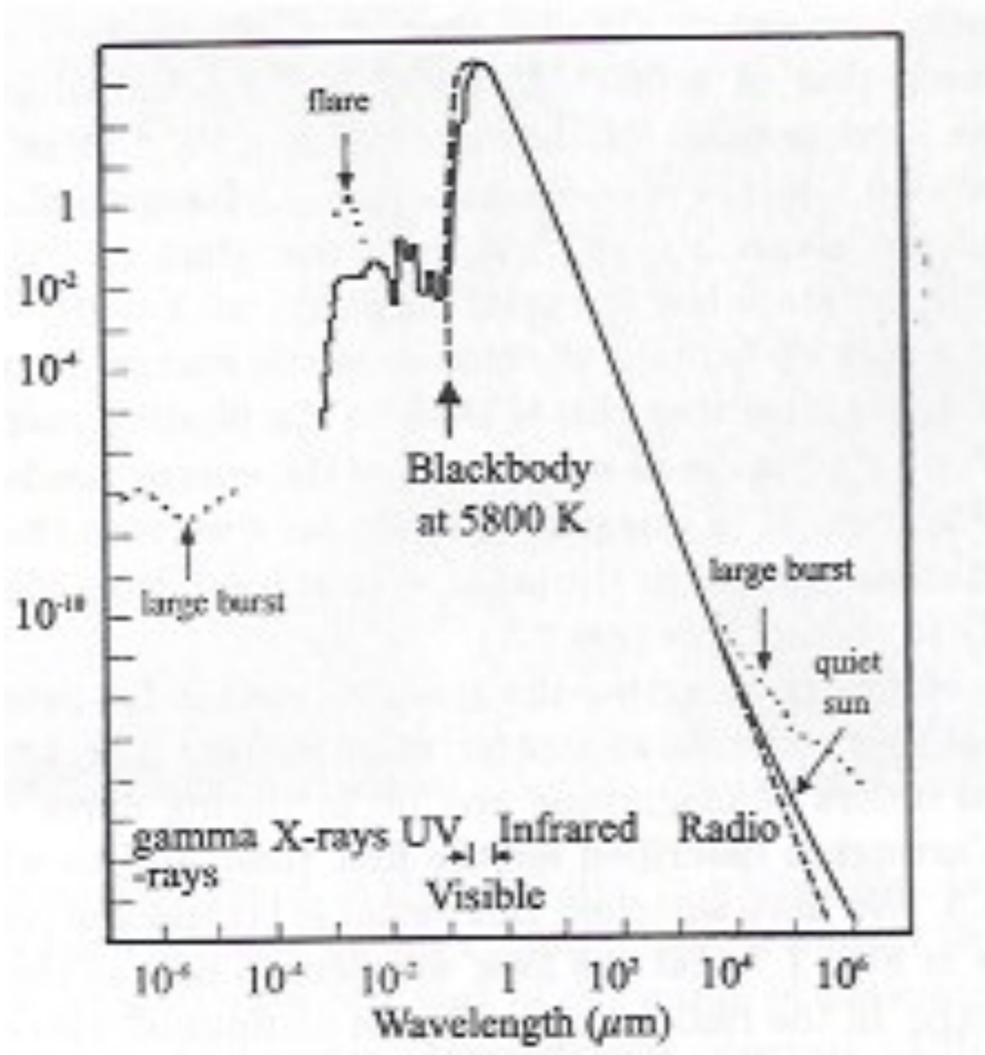
$$B_0(\omega, T) = \frac{\hbar\omega^3}{8\pi^3 c^2} \frac{1}{e^{\hbar\omega/\kappa T} - 1}$$

In the presence of plasma [e.g., Bekefi, 1966]

$$B(\omega, T) = B_0(\omega, T) \left(1 - \frac{\omega_p^2}{\omega^2}\right)$$


Spectrum
vacuum
With plasma
wavelength
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{-18}
10^{-20}
10^{-8}
10^{-7}
10^{-6}
10^{-5}
10^{-4}
10^{-3}

Spectrum of solar radiation



Theory of background radiation

$$\frac{\partial}{\partial t}\frac{I_{\mathbf{k}}^{\sigma T}}{2} = \sum_{\sigma',\sigma''=\pm 1}\int d\mathbf{k}' V_{\mathbf{k},\mathbf{k}'}^{TLL}\,\delta(\sigma\omega_{\mathbf{k}}^{T} - \sigma'\omega_{\mathbf{k}'}^{L} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L})$$

$$\times\left(\sigma\omega_{\mathbf{k}}^{T} I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L} - \frac{\sigma'\omega_{\mathbf{k}'}^{L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L} I_{\mathbf{k}}^{\sigma T}}{2} - \frac{\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L} I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma T}}{2}\right)$$

$$+\sum_{\sigma',\sigma''=\pm 1}\int d\mathbf{k}' V_{\mathbf{k},\mathbf{k}'}^{TLS}\,\delta(\sigma\omega_{\mathbf{k}}^{T} - \sigma'\omega_{\mathbf{k}'}^{L} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{S})\left(\frac{\sigma\omega_{\mathbf{k}}^{T} I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S}}{\mu_{\mathbf{k}-\mathbf{k}'}} - \frac{\sigma'\omega_{\mathbf{k}'}^{L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} I_{\mathbf{k}}^{\sigma T}}{2\mu_{\mathbf{k}-\mathbf{k}'}} - \frac{\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L} I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma T}}{2}\right)$$

$$+\sum_{\sigma',\sigma''=\pm 1}\int d\mathbf{k}' V_{\mathbf{k},\mathbf{k}'}^{TTL}\,\delta(\sigma\omega_{\mathbf{k}}^{T} - \sigma'\omega_{\mathbf{k}'}^{T} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L})\left(\frac{\sigma\omega_{\mathbf{k}}^{T} I_{\mathbf{k}'}^{\sigma' T} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' T}}{2} - \frac{\sigma'\omega_{\mathbf{k}'}^{T} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L} I_{\mathbf{k}}^{\sigma T}}{2} - \frac{\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L} I_{\mathbf{k}'}^{\sigma' T} I_{\mathbf{k}}^{\sigma T}}{4}\right)$$

$$+\sum_{\sigma'=\pm 1}\int d\mathbf{k}'\int d\mathbf{v}\, U_{\mathbf{k},\mathbf{k}'}^{T}\,\delta[\sigma\omega_{\mathbf{k}}^{T} - \sigma'\omega_{\mathbf{k}'}^{L} - (\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}]$$

$$\times\left[\frac{ne^2}{\pi\omega_{pe}^2}\left(\sigma\omega_{\mathbf{k}}^{T} I_{\mathbf{k}'}^{\sigma' L} - \frac{\sigma'\omega_{\mathbf{k}'}^{T} I_{\mathbf{k}}^{\sigma T}}{2}\right)(f_e + f_i) + \pi\frac{m_e}{m_i} I_{\mathbf{k}'}^{\sigma' L}\frac{I_{\mathbf{k}}^{\sigma L}}{2}(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial f_i}{\partial \mathbf{v}}\right]$$

$$\frac{\partial}{\partial t}\frac{I_{\mathbf{k}}^{\sigma T}}{2} = 0 = \sum_{\sigma'=\pm1} \int d\mathbf{k}' \int d\mathbf{v} U_{\mathbf{k},\mathbf{k}'}^{T} \delta[\sigma\omega_{\mathbf{k}}^{T} - \sigma'\omega_{\mathbf{k}'}^{L} - (\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}]$$

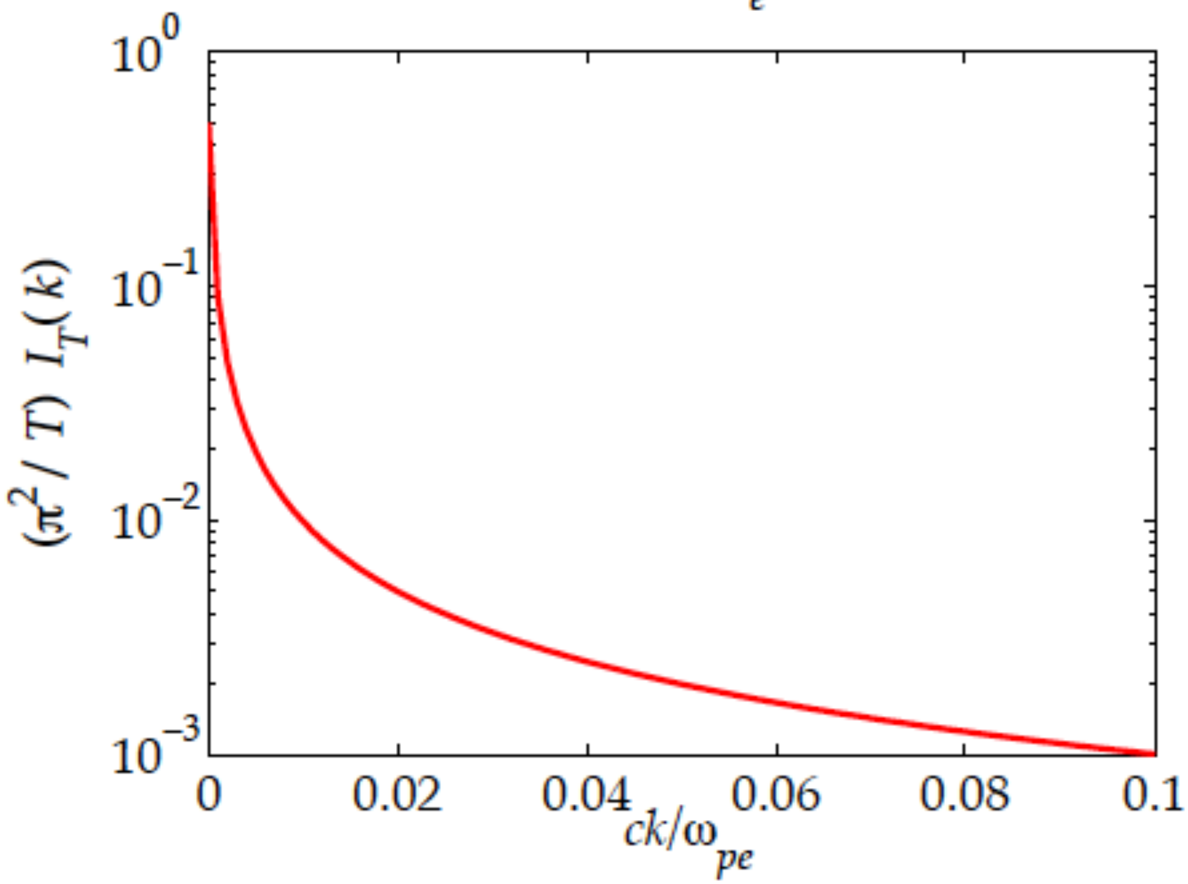
$$\times\left[\frac{ne^2}{\pi\omega_{pe}^2}\left(\sigma\omega_{\mathbf{k}}^{T} I_{\mathbf{k}'}^{\sigma' L} - \frac{\sigma'\omega_{\mathbf{k}'}^{T} I_{\mathbf{k}}^{\sigma T}}{2}\right)(f_e + f_i) + \pi\frac{m_e}{m_i} I_{\mathbf{k}'}^{\sigma' L}\frac{I_{\mathbf{k}}^{\sigma L}}{2}(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial f_i}{\partial \mathbf{v}}\right]$$

↓

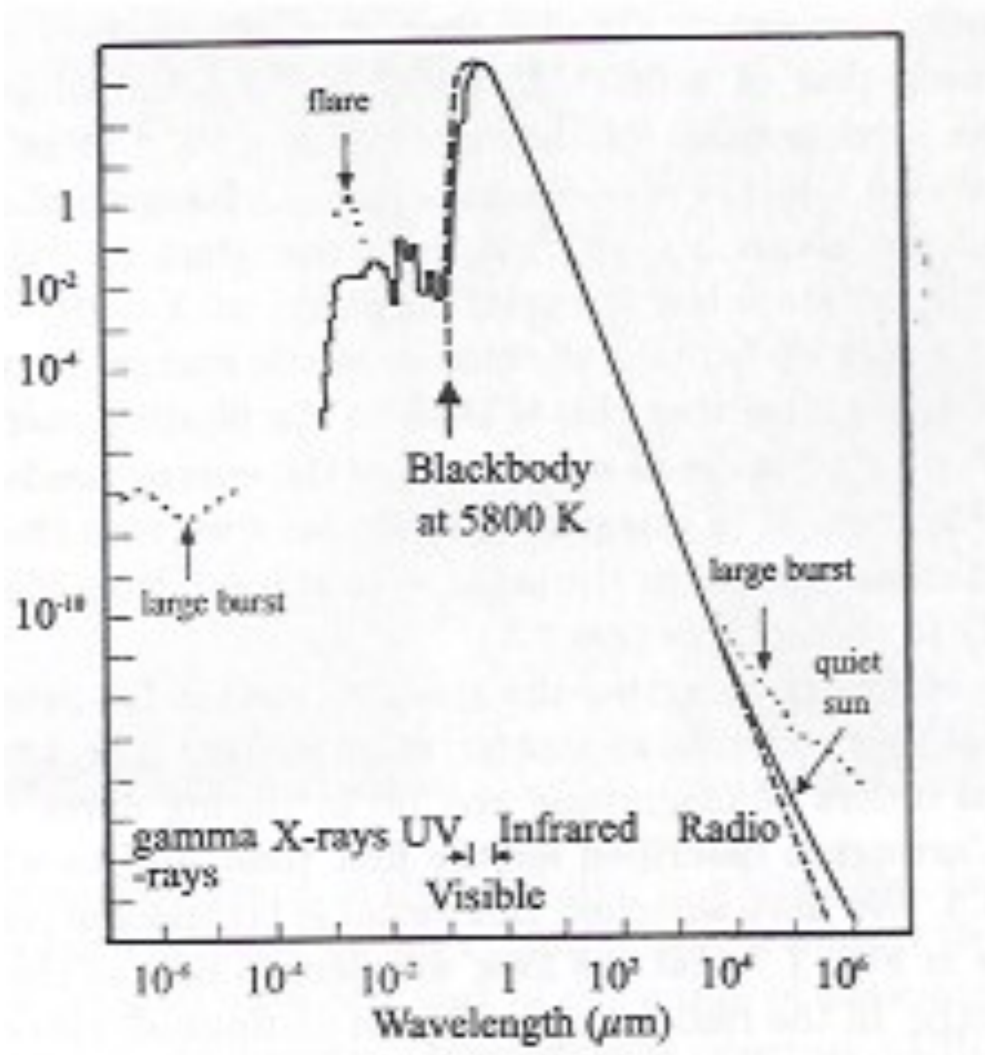
$$0 = \sum_{\sigma=\pm1} \int d\mathbf{k}\, \frac{\mathbf{k}}{k^2}\cdot\frac{\partial}{\partial \mathbf{p}}\sigma\omega_k^L\, \delta(\sigma\omega_k^L - \mathbf{k}\cdot\mathbf{v})\left(\frac{T}{4\pi^2} - I_k^L\right) F_e(p).$$

$$
\begin{aligned}
I_T(k) &= \frac{T}{\pi^2} \frac{1}{1+\sqrt{\dfrac{\omega_{pe}^2 + c^2 k_*^2}{\omega_{pe}^2 + c^2 k^2}}} = \frac{T}{\pi^2} \frac{1}{1+\rho(\kappa)}, \\
\rho(\kappa) &= \sqrt{\frac{1+\alpha^2\,\kappa^2}{1+\kappa^2}}, \\
\kappa &= \frac{ck}{\omega_{pe}}, \qquad \alpha^2 = \frac{2}{3}\frac{c^2}{v_e^2}.
\end{aligned}
$$

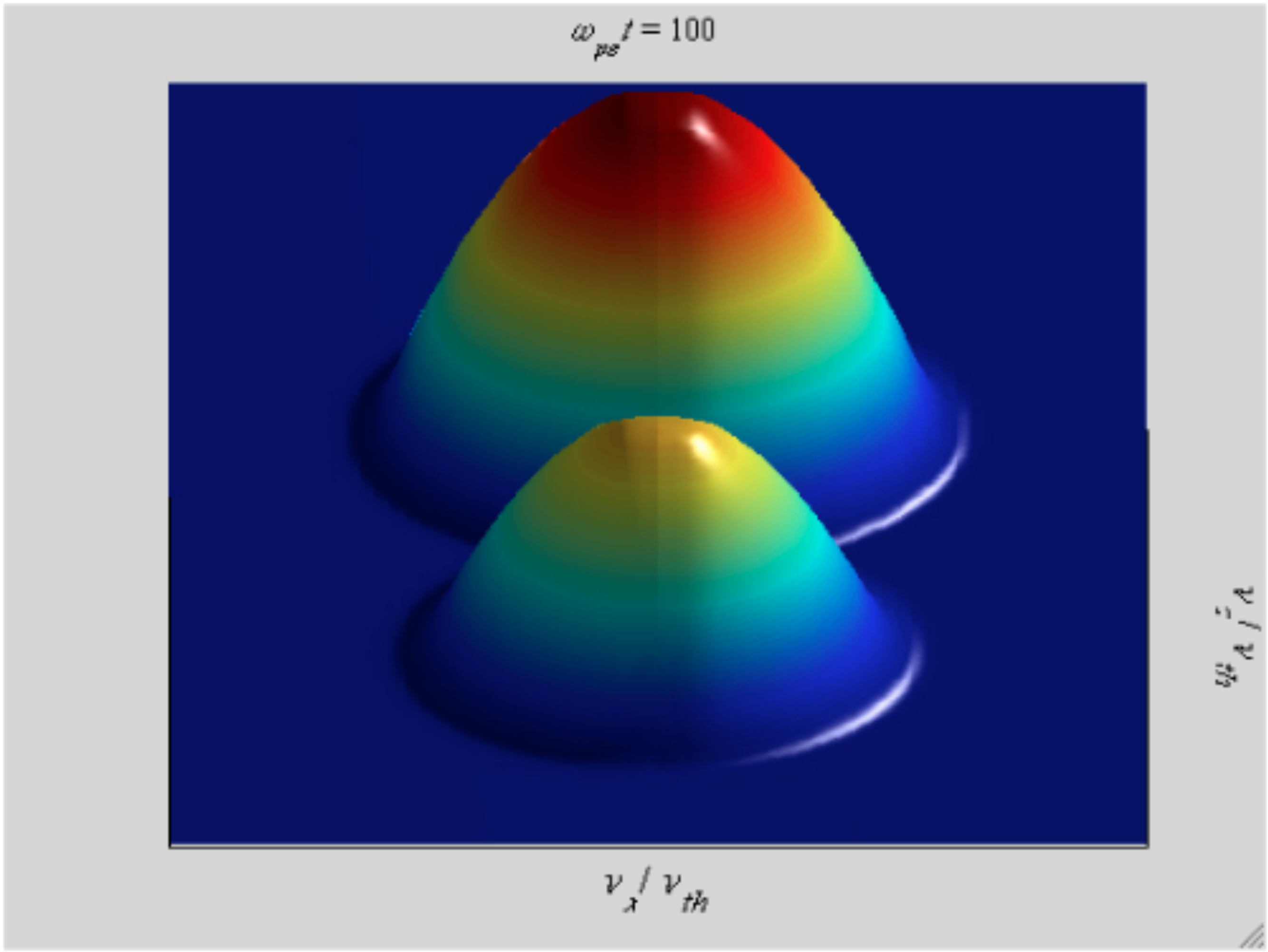
$\alpha^2 = (2/3)(c/v_e)^2 = 10^8$



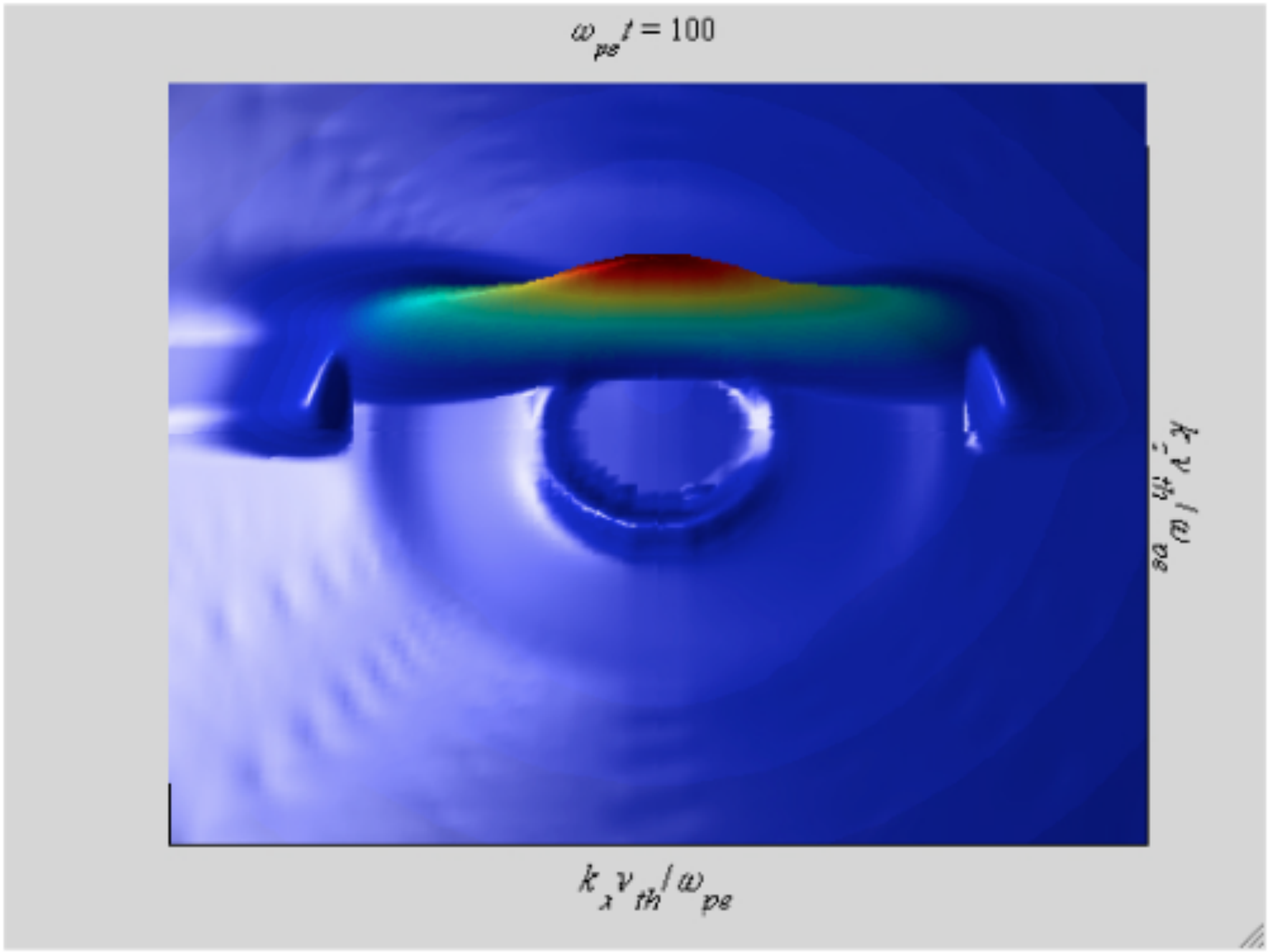
Spectrum of solar radiation



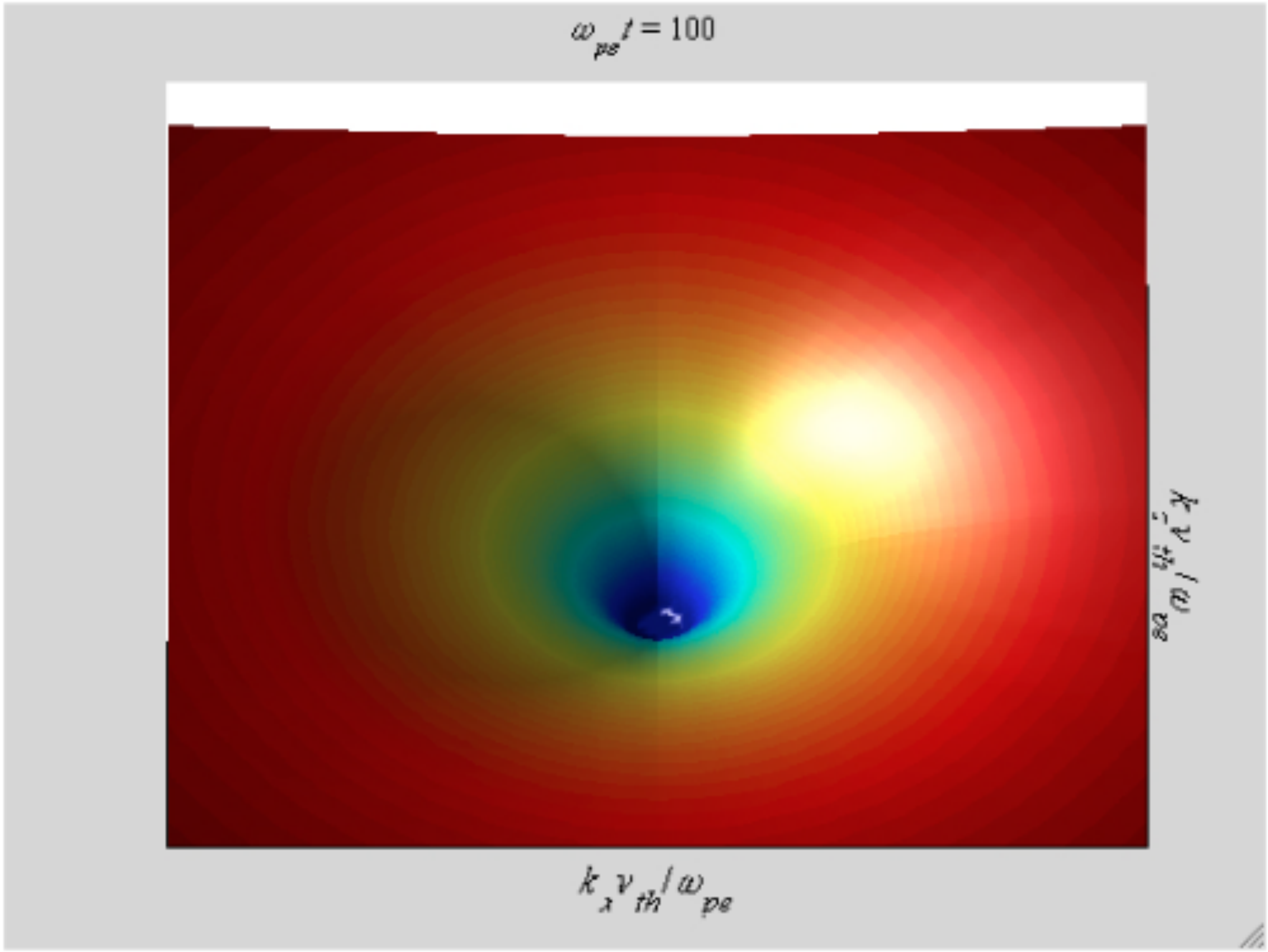
Electron Velocity Distribution Function



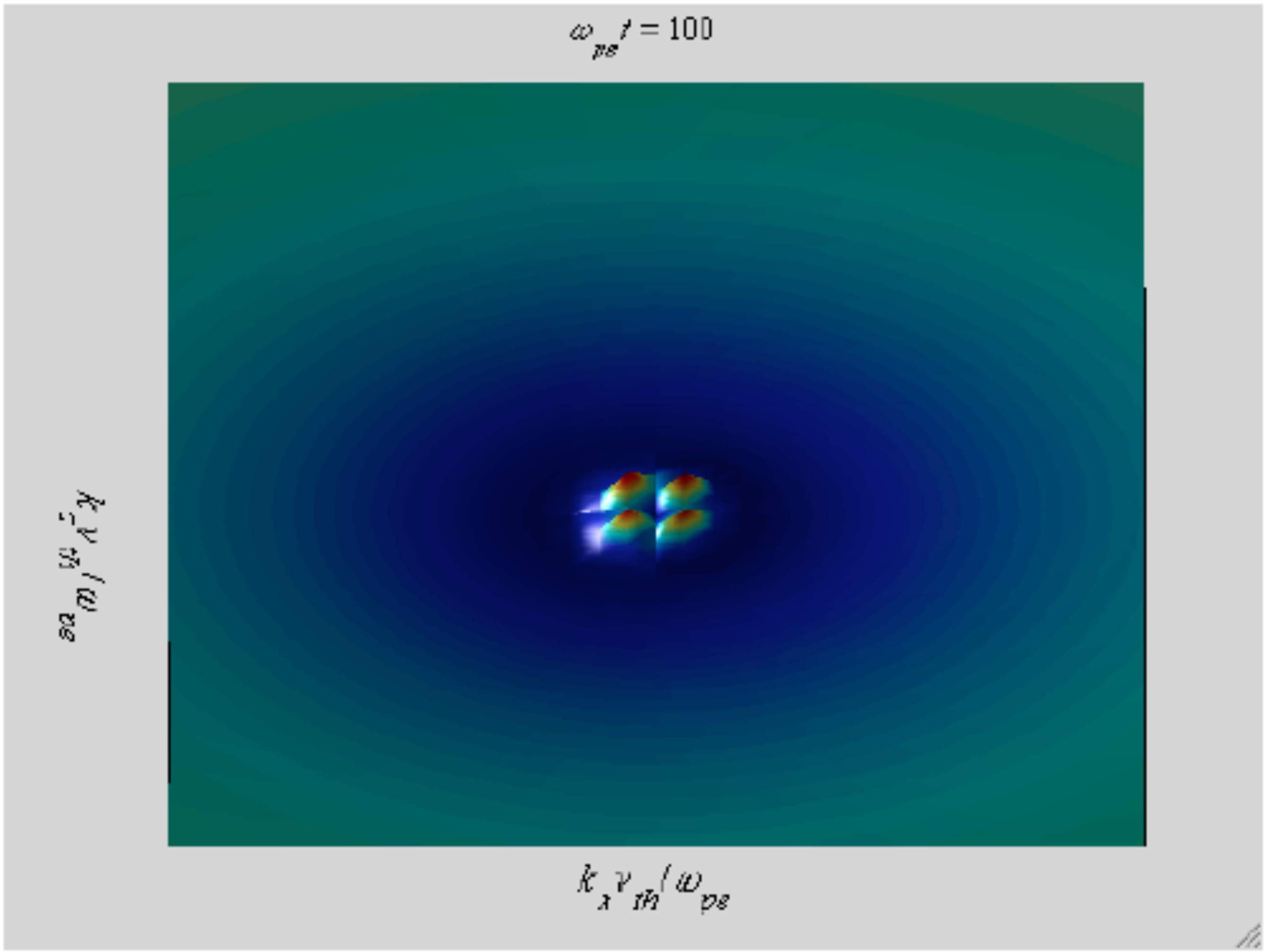
Langmuir Turbulence Spectrum



Ion-Acoustic Turbulence Spectrum



Electromagnetic Radiation Spectrum



Part 2. Conclusions and Discussion

- Despite 6 decades of research, the plasma emission was not understood in complete quantitative terms until now.

- We solved EM weak turbulence theory to demonstrate plasma emission for the first time and compared against few available 2D EM PIC simulation results.

General Conclusion

- Perturbative nonlinear kinetic theory called "weak turbulence theory" successfully explains many facets of space plasma phenomena – electron kappa VDF, plasma emission, …
- Basic methodology can be extended for other applications, most notably, turbulent phenomena in *magnetized* plasmas.