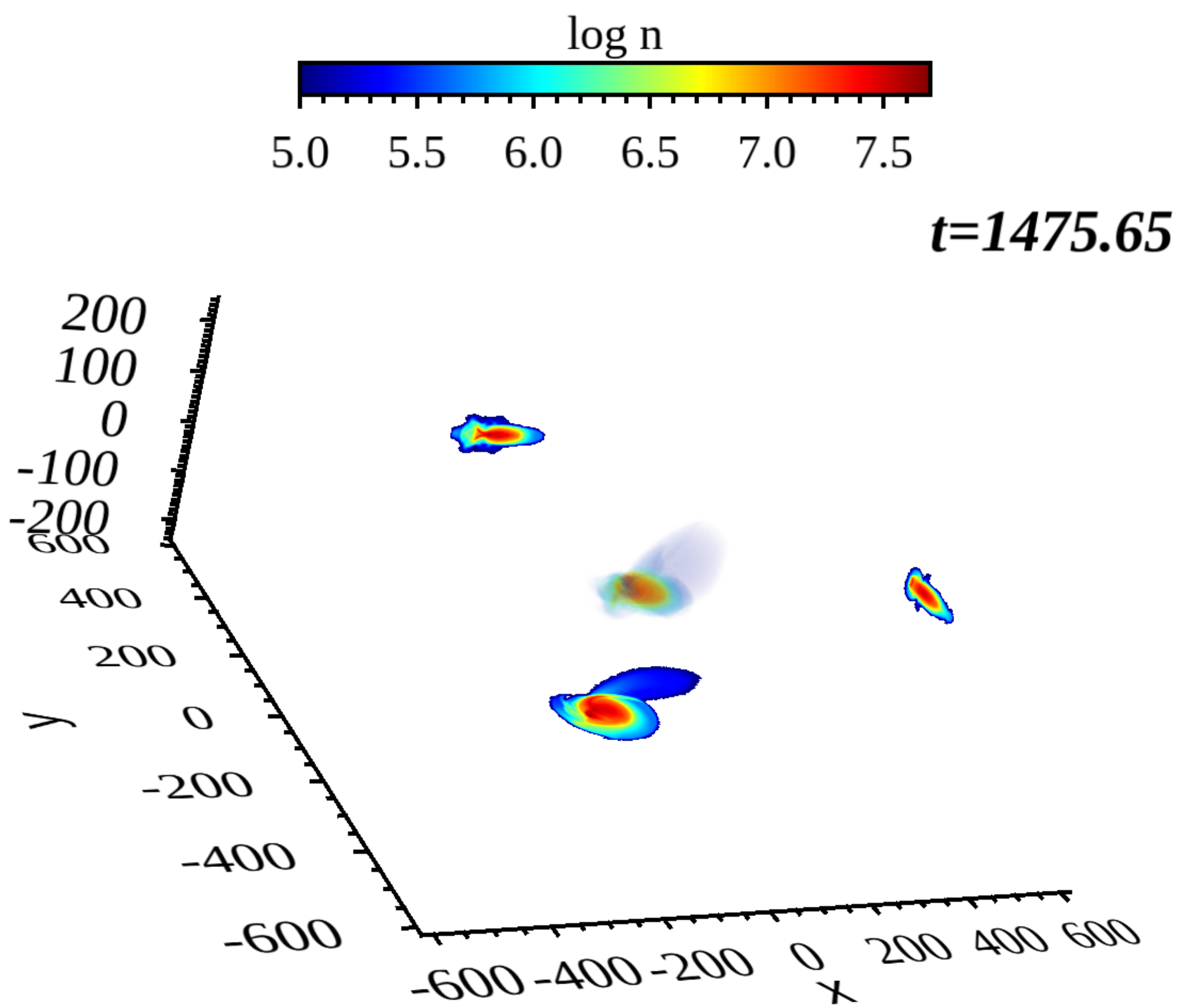
A Proper Discretization of Hydrodynamic Equations in Cylindrical Coordinates for Astrophysical Simulations

Tomoyuki Hanawa (Chiba U.)



Episodic gas accretion onto a protostar with gas disk

cf. Dullemond+19, Kueffimier+19

Cloudlet capture by a young star

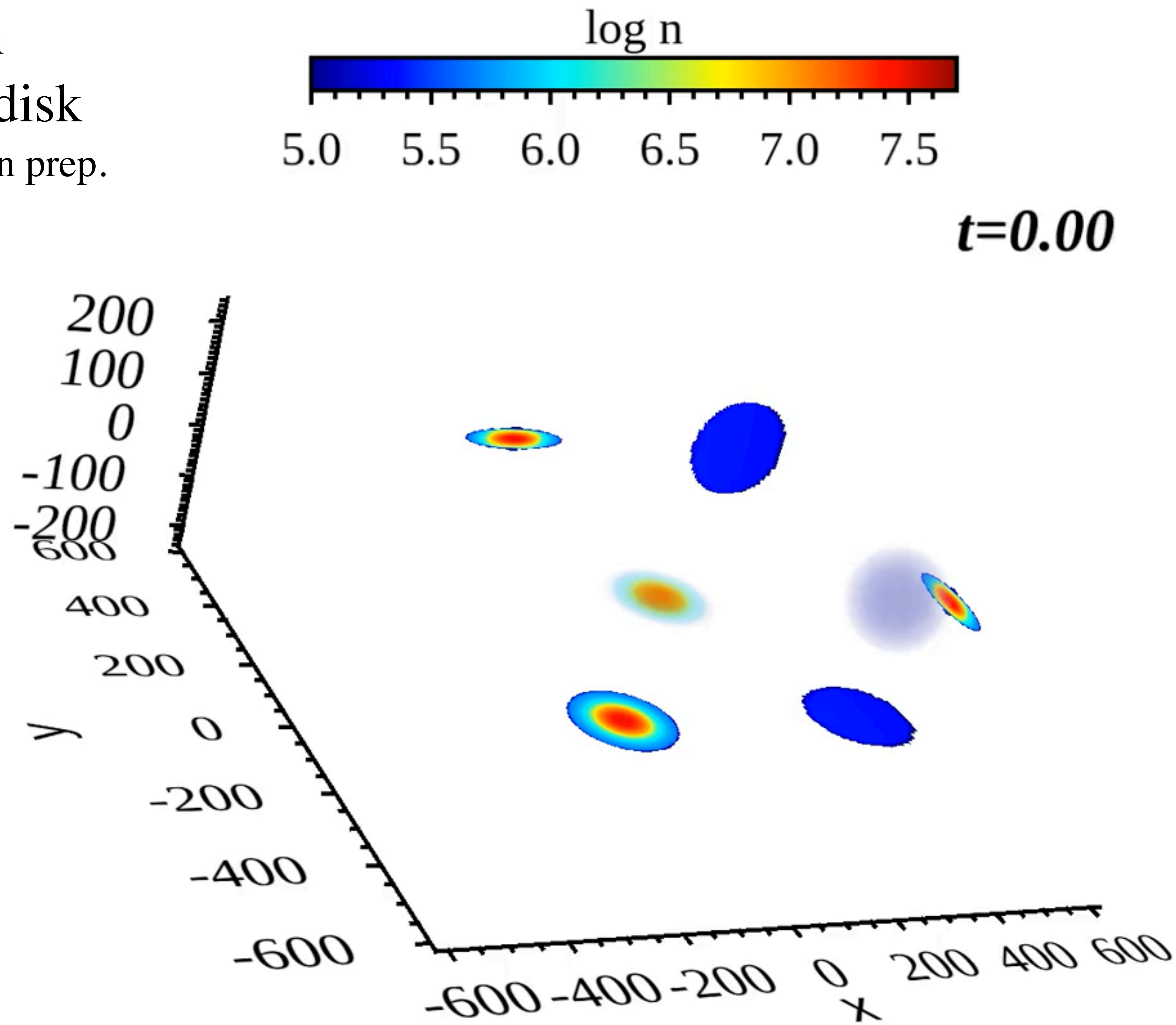
3D simulation on the Cylindrical Grid, (r, φ, z)

Free-stream preservation in addition to mass, angular momentum & energy conservation

Tomoyuki Hanawa & Yosuke Matsumoto (Chiba U.)   ApJ, 907, 43 (2021)

Cloudlet accretion
onto a protostar with disk

Hanawa, Sakai, & Yamamoto in prep.



Cylindrical coordinates

$(\rho, v_r, v_\varphi, v_z, P)$

Advantages: easy to handle (1) angular momentum conservation, and (2) axisymmetric equilibrium (disk & torus)

Problems in the Cylindrical Coordinates, (r, φ, z).

1. Apparently Large Centrifugal Force around $r = 0$

$$
\begin{aligned}
\boldsymbol{a} &= \left(\ddot{r} - r\dot{\varphi}^2\right)\boldsymbol{e}_r + \left(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}\right)\boldsymbol{e}_\varphi + \ddot{z}\boldsymbol{e}_z \\
&= \left(\dot{v}_r - \frac{v_\varphi^2}{r}\right)\boldsymbol{e}_r + \left(\dot{v}_\varphi + \frac{v_r v_\varphi}{r}\right)\boldsymbol{e}_\varphi + \dot{v}_z\boldsymbol{e}_z \\
&= \left(\dot{v}_r - \frac{j^2}{r^3}\right)\boldsymbol{e}_r + \frac{\dot{j}}{r}\boldsymbol{e}_\varphi + \dot{v}_z\boldsymbol{e}_z
\end{aligned}
$$

2. Tight CFL condition ($\Delta t < r\Delta\varphi/|v+c|$) around $r = 0$ if $\Delta\varphi$ is fixed.

3. Uniform flow, $(\rho_0, \boldsymbol{v}, P_0)$, is an exact solution of the differential equations but not of the discretized form in (r, φ, z).
(**Not** free-stream preserving)

We have removed these technical problems.
Our method is based on the finite volume scheme.

Hydrodynamic Equations in the conservation form

Mass, (angular) momentum, and energy conservation

$$\frac{\partial}{\partial t}\begin{bmatrix} r\rho \\ r\rho v_r \\ r^2\rho v_\varphi \\ r\rho v_z \\ r\rho E \end{bmatrix} + \frac{\partial}{\partial r}\begin{bmatrix} r\rho v_r \\ r\left(\rho {v_r}^2 + P\right) \\ r^2\rho v_r v_\varphi \\ r\rho v_r v_z \\ r\rho v_r H \end{bmatrix} + \frac{\partial}{\partial \varphi}\begin{bmatrix} \rho v_\varphi \\ \rho v_\varphi v_r \\ r\left(\rho {v_\varphi}^2 + P\right) \\ \rho v_\varphi v_z \\ \rho v_\varphi H \end{bmatrix}$$

$$+ \frac{\partial}{\partial z}\begin{bmatrix} r\rho v_z \\ r\rho v_z v_r \\ r^2\rho v_z v_\varphi \\ \rho v_\varphi v_z \\ \rho v_\varphi H \end{bmatrix} = \begin{bmatrix} 0 \\ r\rho g_r + \rho {v_\varphi}^2 + P \\ r^2\rho g_\varphi \\ r\rho g_z \\ r\rho \boldsymbol{g}\cdot\boldsymbol{v} \end{bmatrix}$$

centrifugal force

$$\frac{\partial U}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{F} = 0$$

volume integral

$dV = r\, dr\, d\varphi\, dz$

$$\int \frac{\partial U}{\partial t} dV + \int \boldsymbol{F}\cdot d\boldsymbol{S} = 0$$

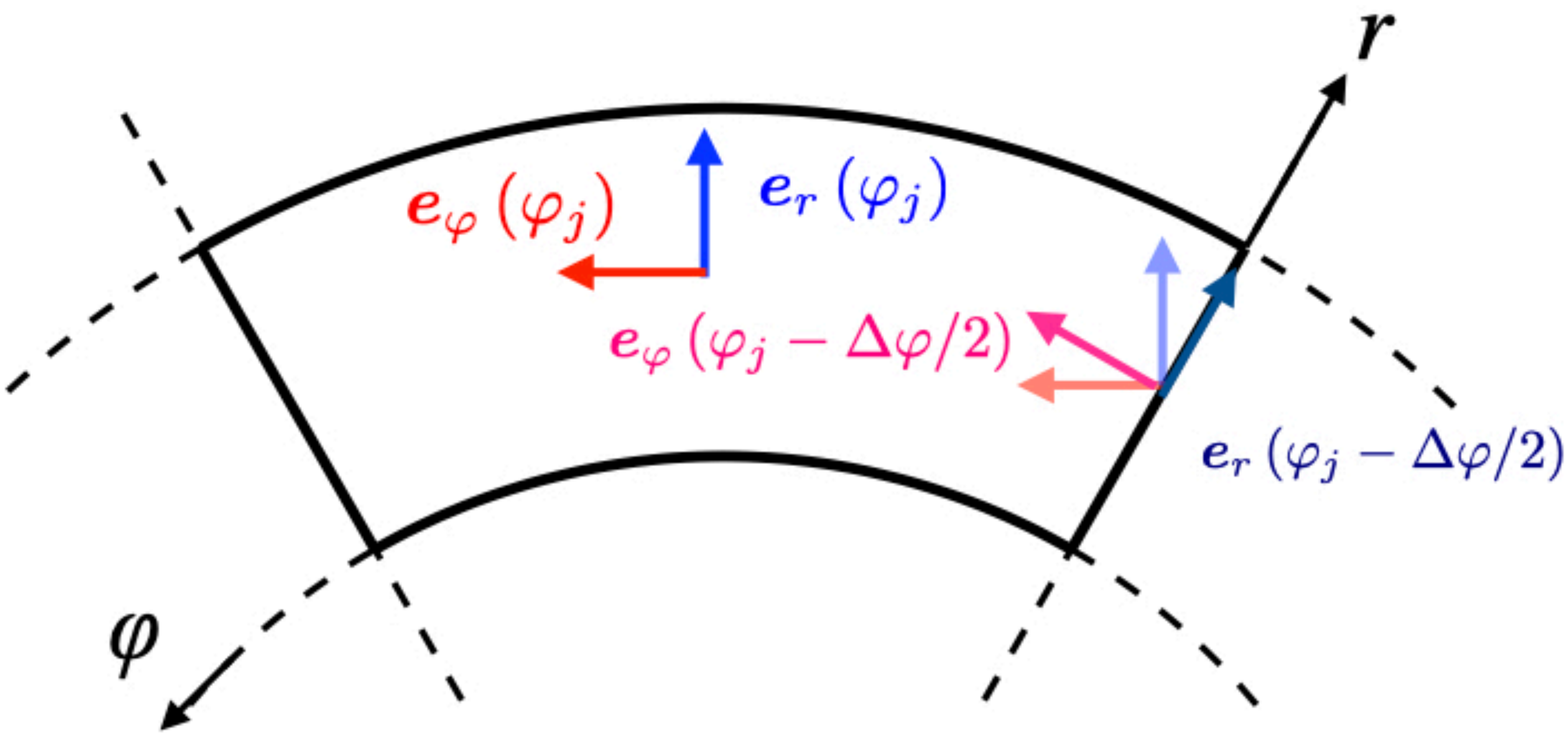
Vector

$$\frac{\partial}{\partial t}(\rho\boldsymbol{v}) + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}\boldsymbol{v} + P\boldsymbol{I}) = \rho\boldsymbol{g},$$

$$\frac{\partial}{\partial t}\iiint_{V_j}\rho\boldsymbol{v}dV + \iint_{\partial V_j}(\rho\boldsymbol{v}\boldsymbol{v} + P\boldsymbol{I})\cdot d\boldsymbol{S} = \iint_{v_j}\rho\boldsymbol{g}dV.$$

Caution in the volume integral:

We should appreciate the fact that the unit vectors, $\boldsymbol{e}_r$ and $\boldsymbol{e}_\varphi$, on the azimuthal cell surface are different from those at the cell center.

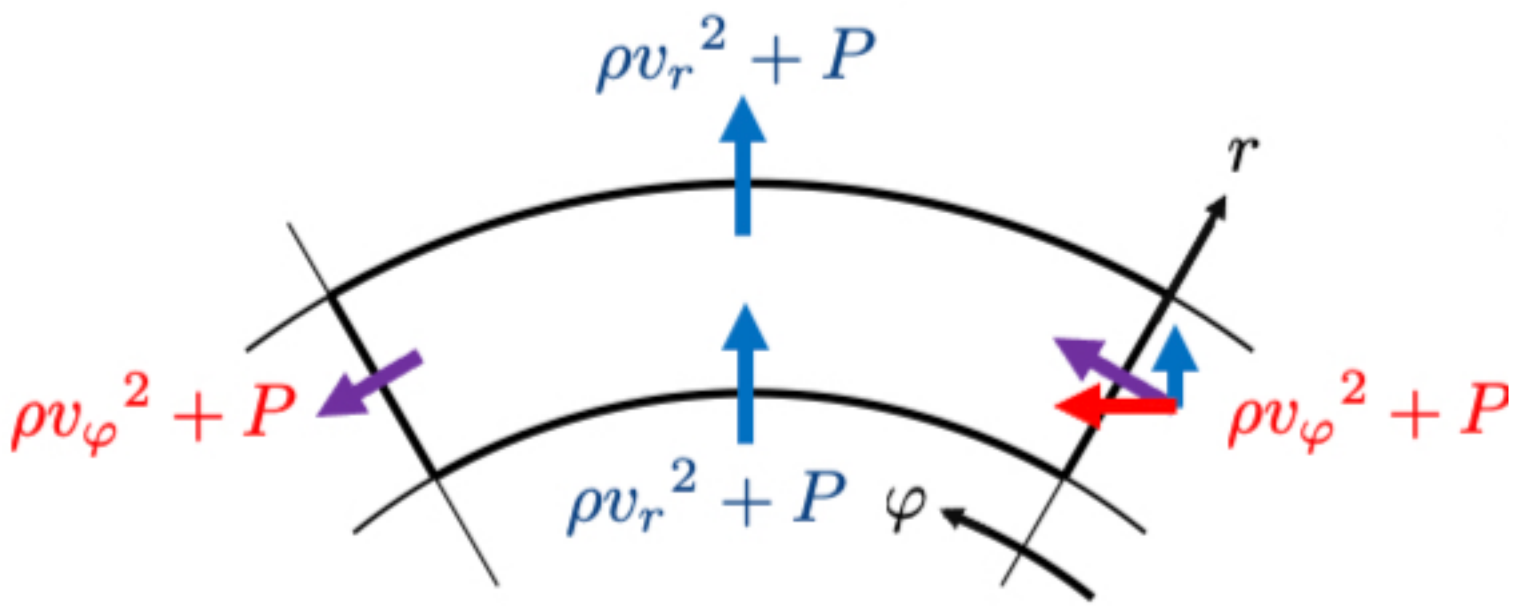


$$\boldsymbol{e}_r(\varphi_j \pm \Delta\varphi/2) = \cos\frac{\Delta\varphi}{2}\boldsymbol{e}_r(\varphi_j) \mp \sin\frac{\Delta\varphi}{2}\boldsymbol{e}_\varphi(\varphi_j)$$

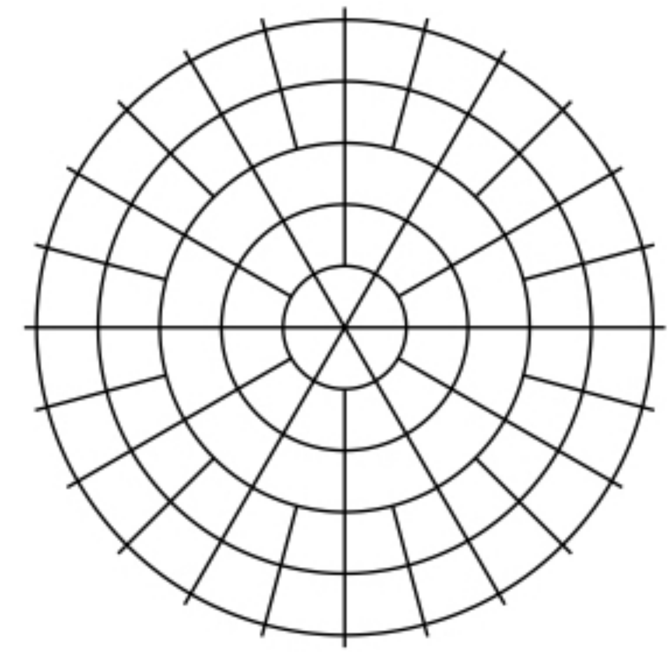
$$\boldsymbol{v}_{j-1/2} = \left(v_{r,j-1/2}\cos\frac{\Delta\varphi}{2} + v_{\varphi,j-1/2}\sin\frac{\Delta\varphi}{2}\right)\boldsymbol{e}_{r,j} + \left(-v_{r,j-1/2}\sin\frac{\Delta\varphi}{2} + v_{\varphi,j-1/2}\cos\frac{\Delta\varphi}{2}\right)\boldsymbol{e}_{\varphi,j}$$

4 Improvements in the numerical method

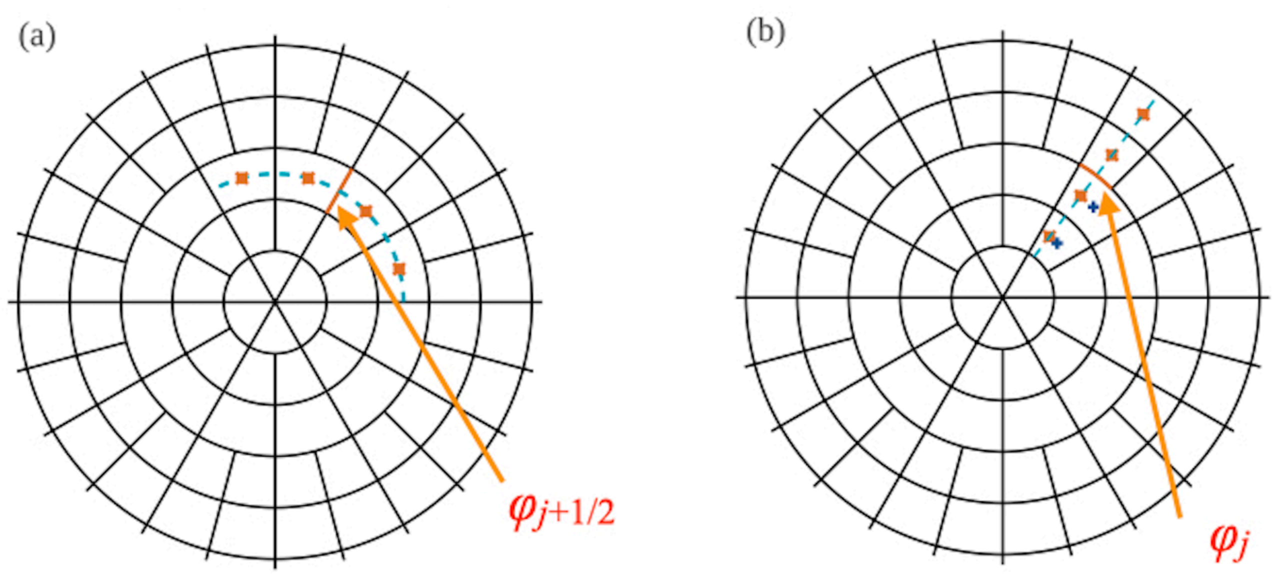
(1) Evaluating the centrifugal force on the azimuthal cell surface (not at the cell center).



(2) Coarse grid near the origin
cf. AMR $r\Delta\varphi \simeq \Delta r$



(3) Interpolating v_n and v_t instead of v_r and v_φ.

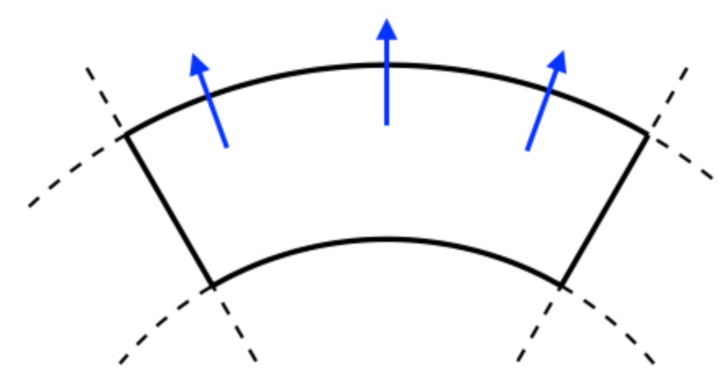


$v_\mathrm{n} = \boldsymbol{v} \cdot \boldsymbol{e}_\varphi(\varphi_{j+1/2})$
$v_\mathrm{t} = \boldsymbol{v} \cdot \boldsymbol{e}_r(\varphi_{j+1/2})$

$v_\mathrm{n} = \boldsymbol{v} \cdot \boldsymbol{e}_r(\varphi_j)$
$v_\mathrm{t} = \boldsymbol{v} \cdot \boldsymbol{e}_\varphi(\varphi_j)$

(4) Correction factors for surface area

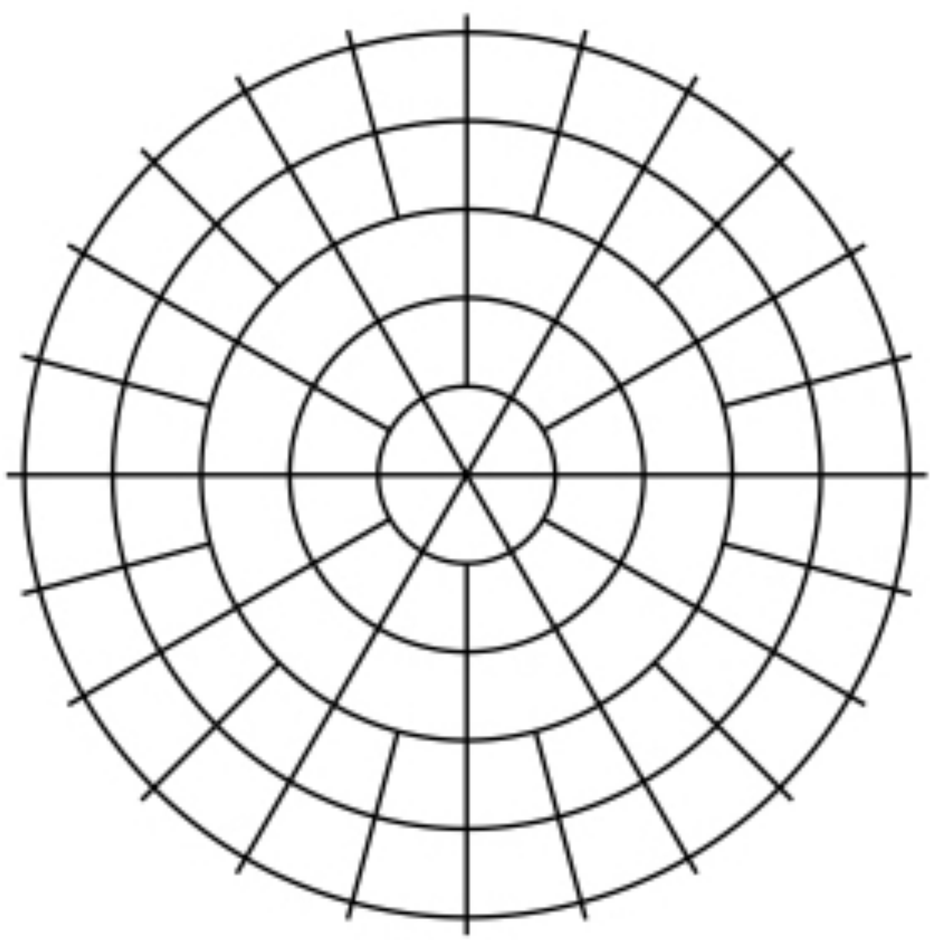
$$\Delta \boldsymbol{S}_{i+1/2} \cdot \boldsymbol{e}_r = f_{i+1/2} r_{i+1/2} \Delta\varphi$$



$$\int_{\varphi_j - \Delta\varphi/2}^{\varphi_j + \Delta\varphi/2} \boldsymbol{e}_r(\varphi)\, d\varphi \neq \boldsymbol{e}_r(\varphi_j) \Delta\varphi$$

Surface Area Correction Factors

Cell Type	(A)	(B)	(C)	
f_ρ	$\frac{\Delta\varphi}{2}\left(\sin\frac{\Delta\varphi}{2}\right)^{-1}$	$\frac{\Delta\varphi}{2}\left(\sin\frac{\Delta\varphi}{2}\right)^{-1}\left[1-\frac{2r_{i+1/2}}{\Delta r_i}\sin^2\frac{\Delta\varphi}{8}\right]$	$\frac{\Delta\varphi}{4}\left(\tan\frac{\Delta\varphi}{4}\right)^{-1}$	mass, energy, v_z
f_r	$\left(\frac{\Delta\varphi}{2}\right)\left(\tan\frac{\Delta\varphi}{2}\right)^{-1}$	$\frac{\Delta\varphi}{\sin\Delta\varphi}\left(1-\frac{r_{i+1/2}+\Delta r_i}{\Delta r_i}\sin^2\frac{\Delta\varphi}{4}\right)$	$\frac{\Delta\varphi}{\sin\Delta\varphi}\left[1-2\sin^2\frac{\Delta\varphi}{4}\right]$	v_r
f_φ	$\frac{\Delta\varphi}{\sin\Delta\varphi}$	$\frac{\Delta\varphi}{\sin\Delta\varphi}\left[1-\frac{r_{i+1/2}^2}{r_i\Delta r_i}\sin^2\frac{\Delta\varphi}{4}\right]$	$\frac{\Delta\varphi}{2}\left(\sin\frac{\Delta\varphi}{2}\right)^{-1}$	$j = rv_\varphi$



Type (A) $\Delta\varphi_{i-1/2} = \Delta\varphi_i = \Delta\varphi_{i+1/2}$

Type (B) $\Delta\varphi_{i-1/2} = \Delta\varphi_i = 2\Delta\varphi_{i+1/2}$

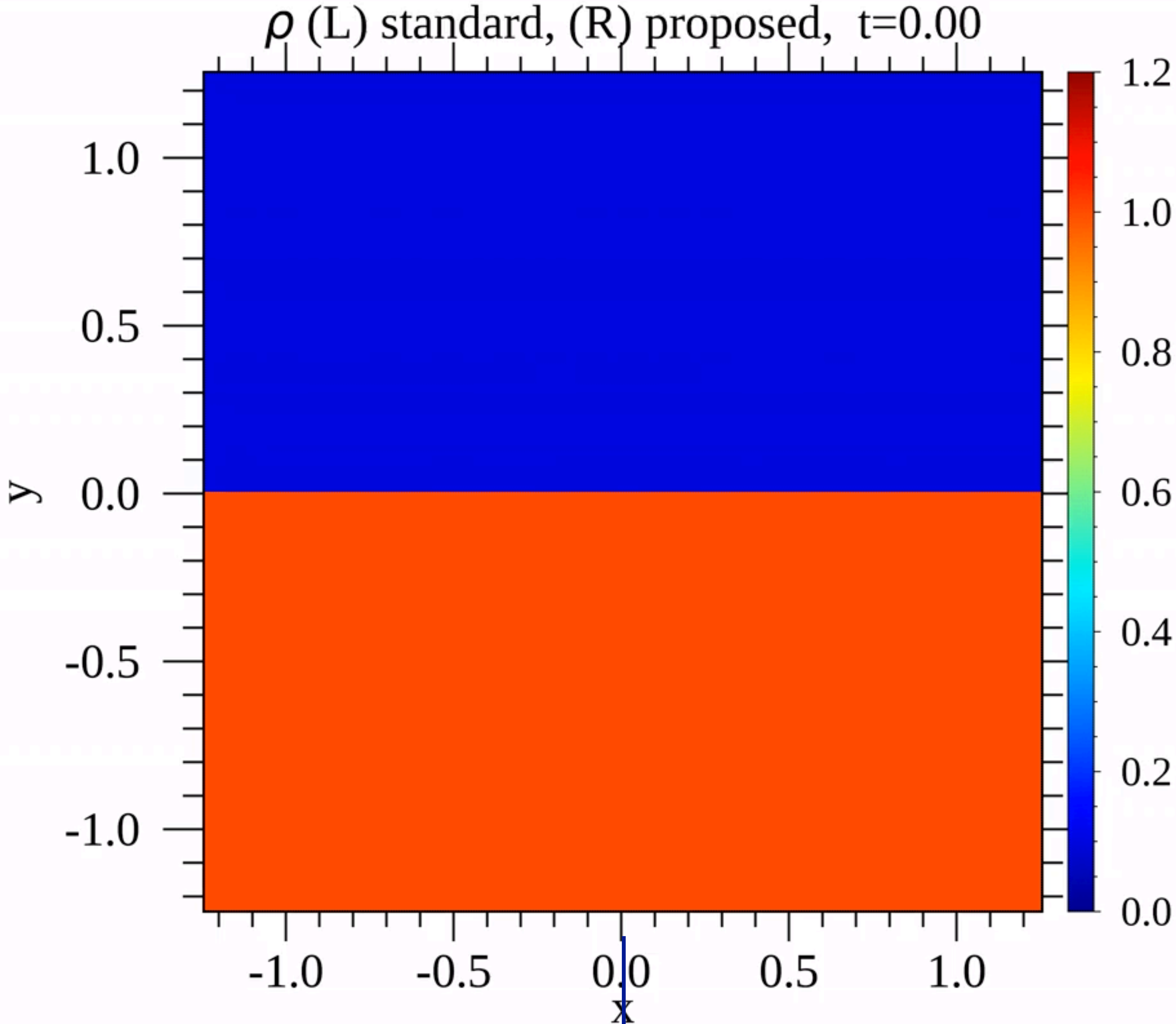
Type (C) $2\Delta\varphi_{i-1/2} = \Delta\varphi_i = 2\Delta\varphi_{i+1/2}$

The correction factors are fixed so that a uniform flow, i.e., constant ρ, $\boldsymbol{v}$, P, is an exact solution of the finite difference equations.

free-stream preserving

Sod Shock Tube Problem (Comparison)

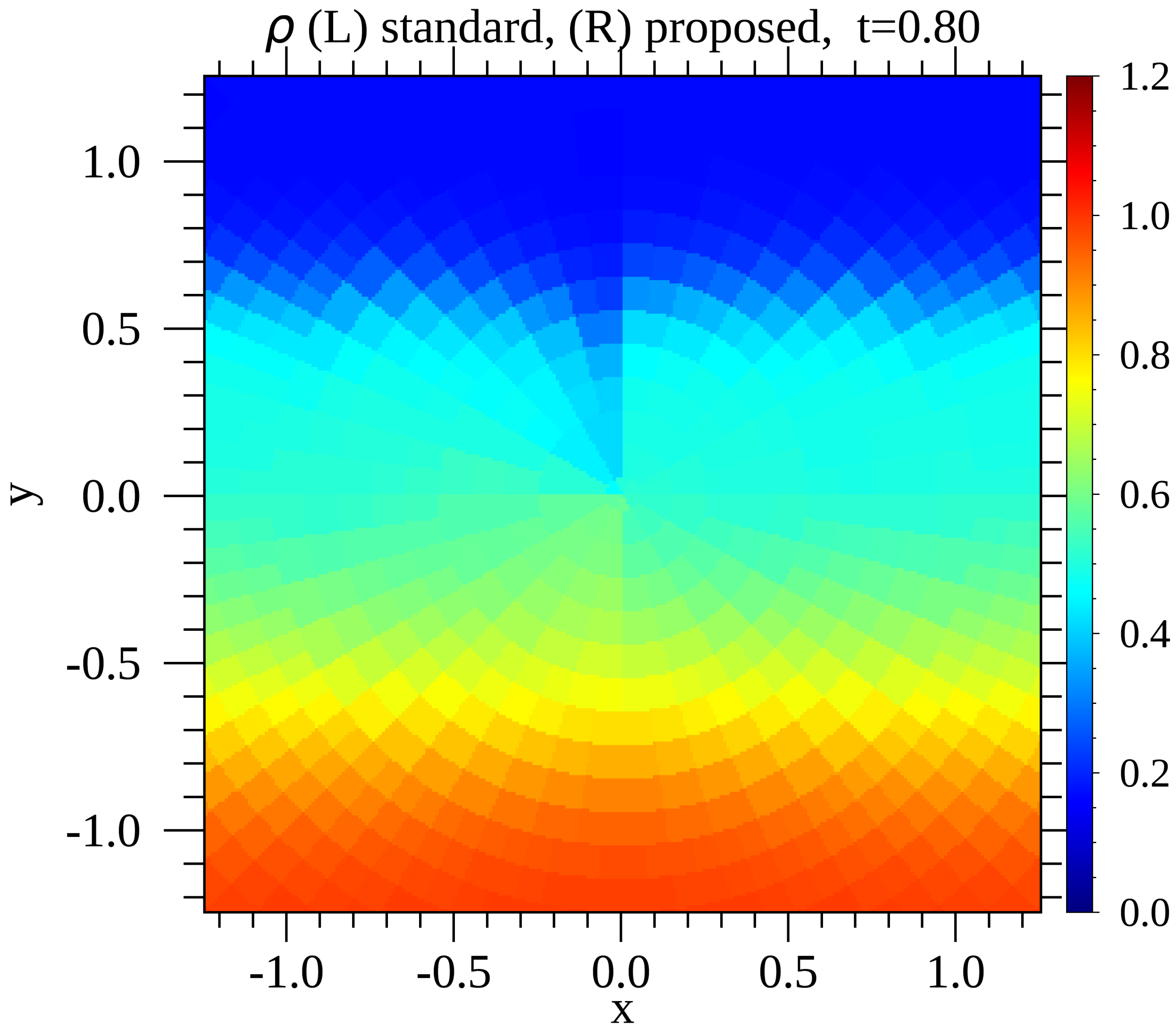
AMR-type variable $\Delta\varphi$

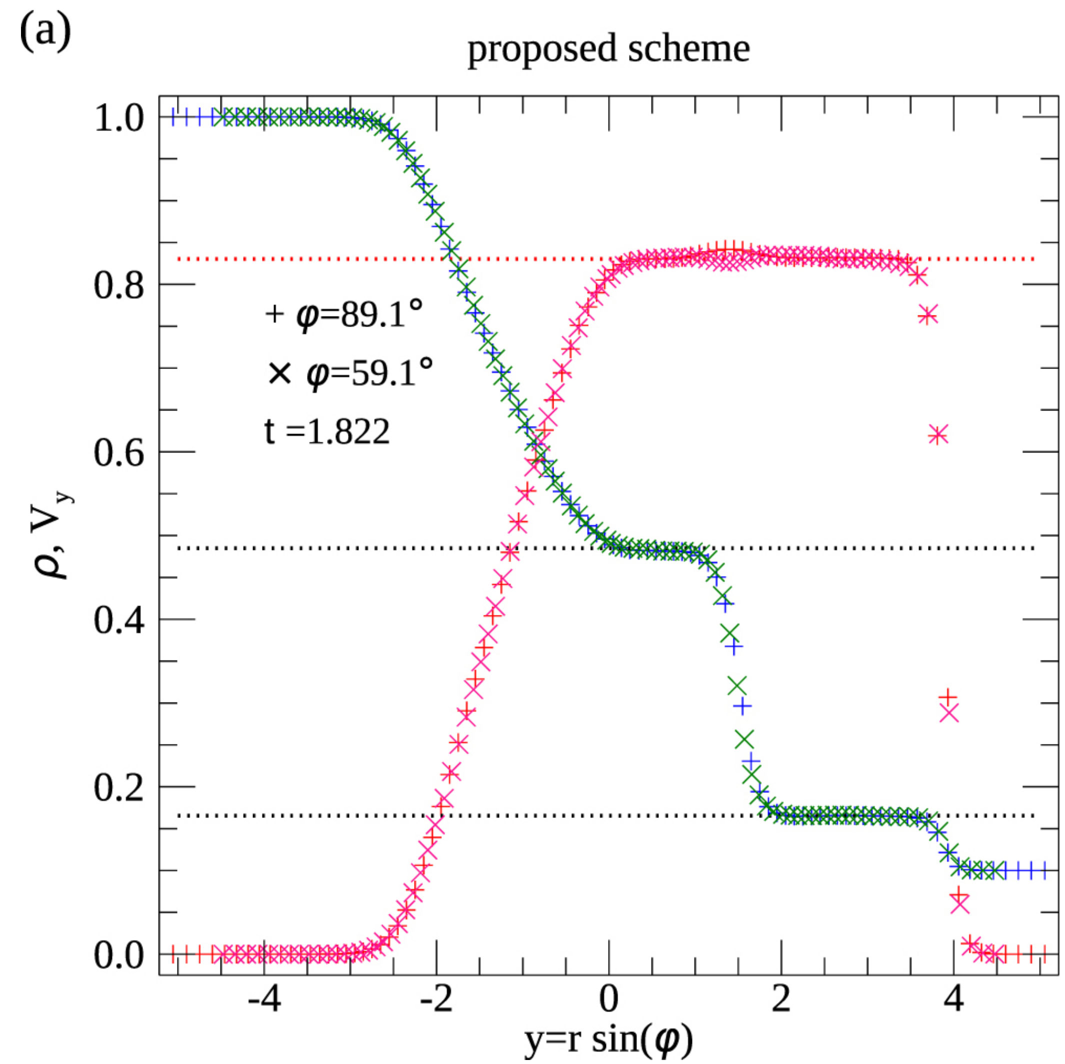


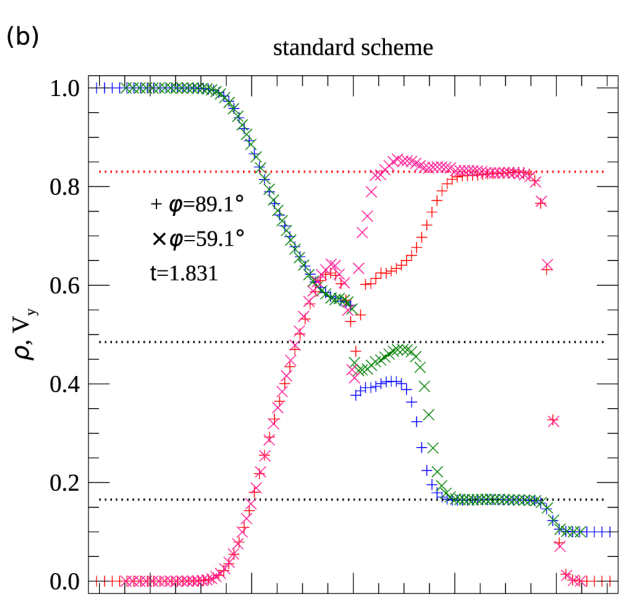
standard
(large error)

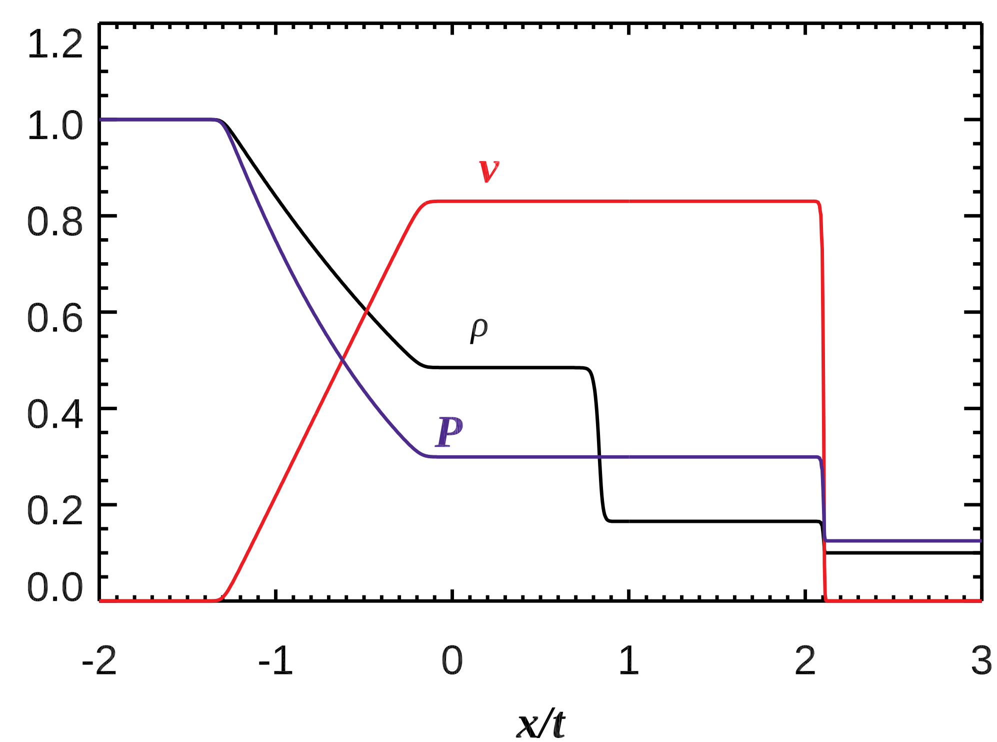
proposed
(good)

AMR-type variable $\Delta\varphi$









The current standard scheme does not work even at $\Delta\varphi = \pi/96$

uniform $\Delta\varphi = \pi/96$ in standard

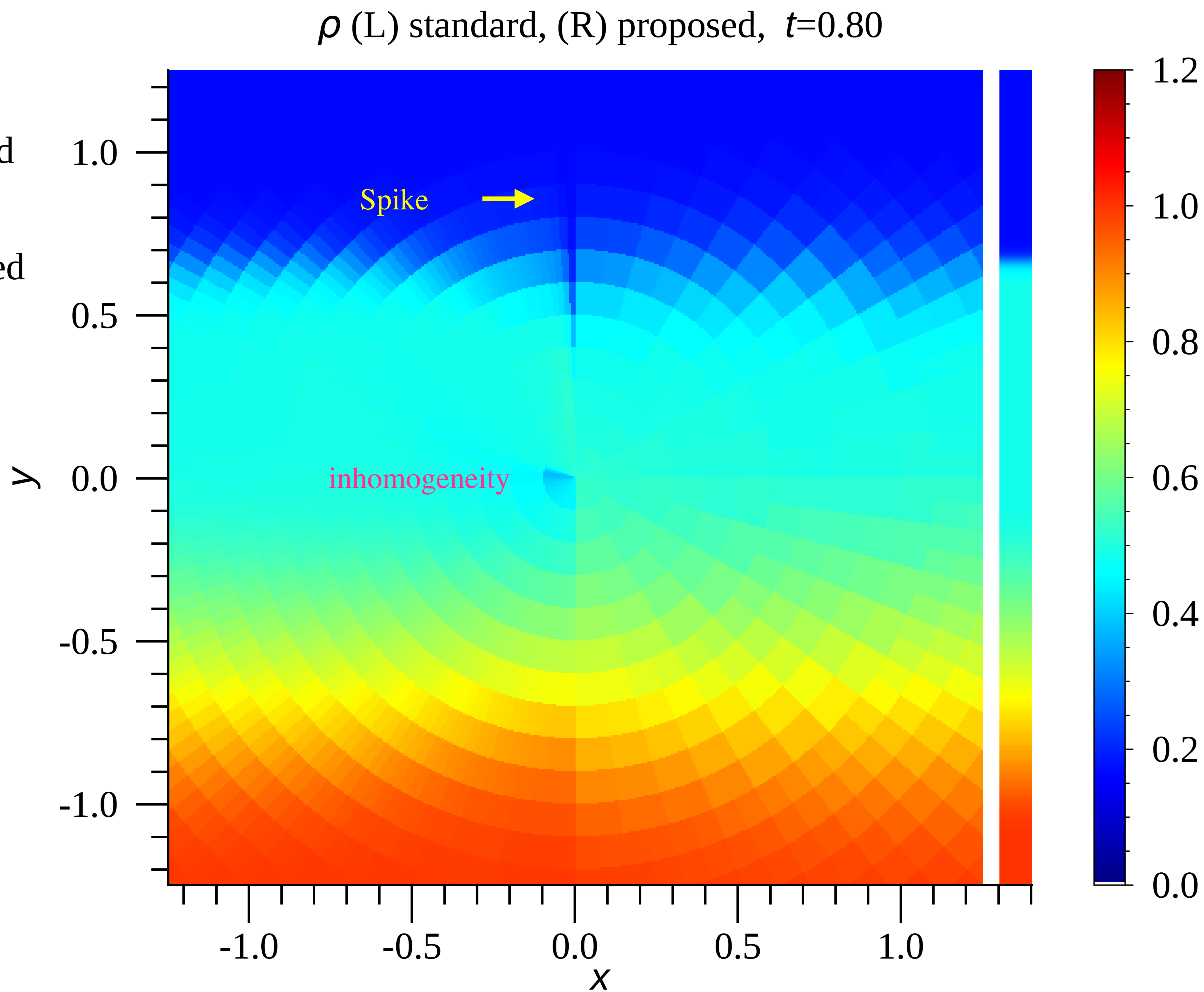
Δr=0.1

Δt=0.00057

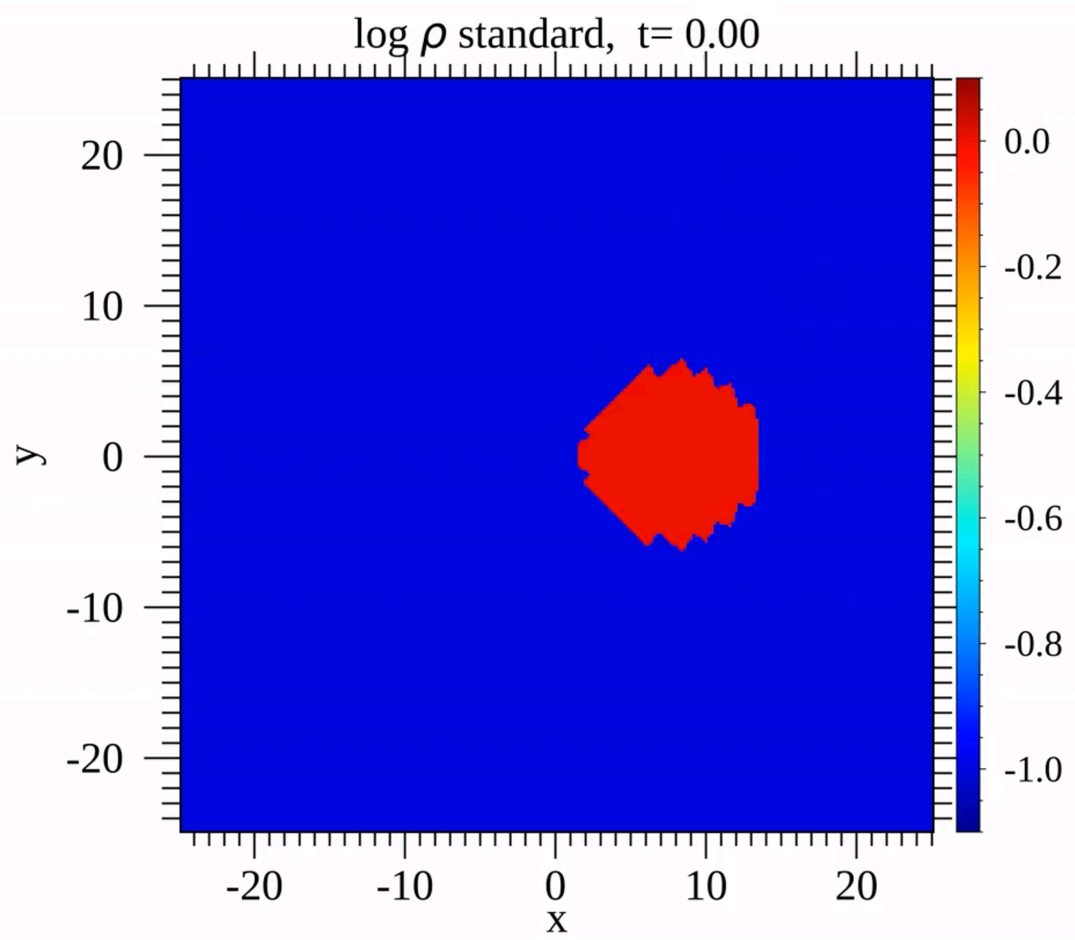
for standard

Δt=0.02

for proposed



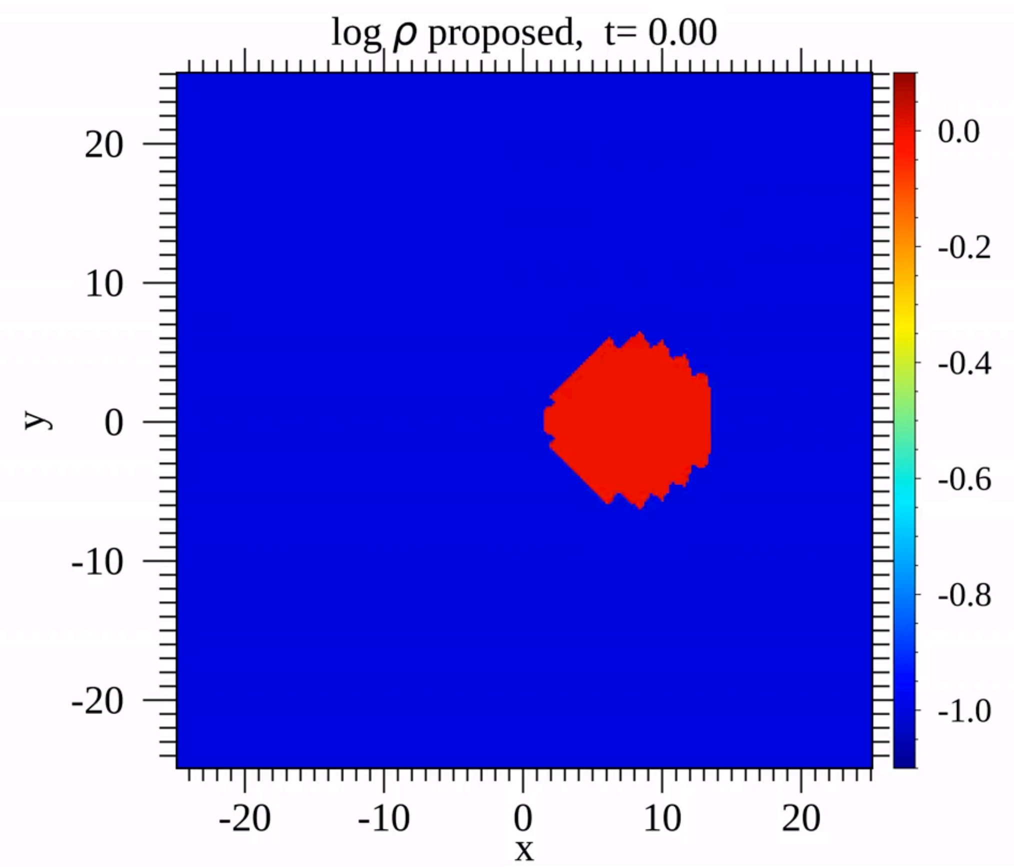
AMR-type variable $\Delta\varphi$



←standard

proposed →

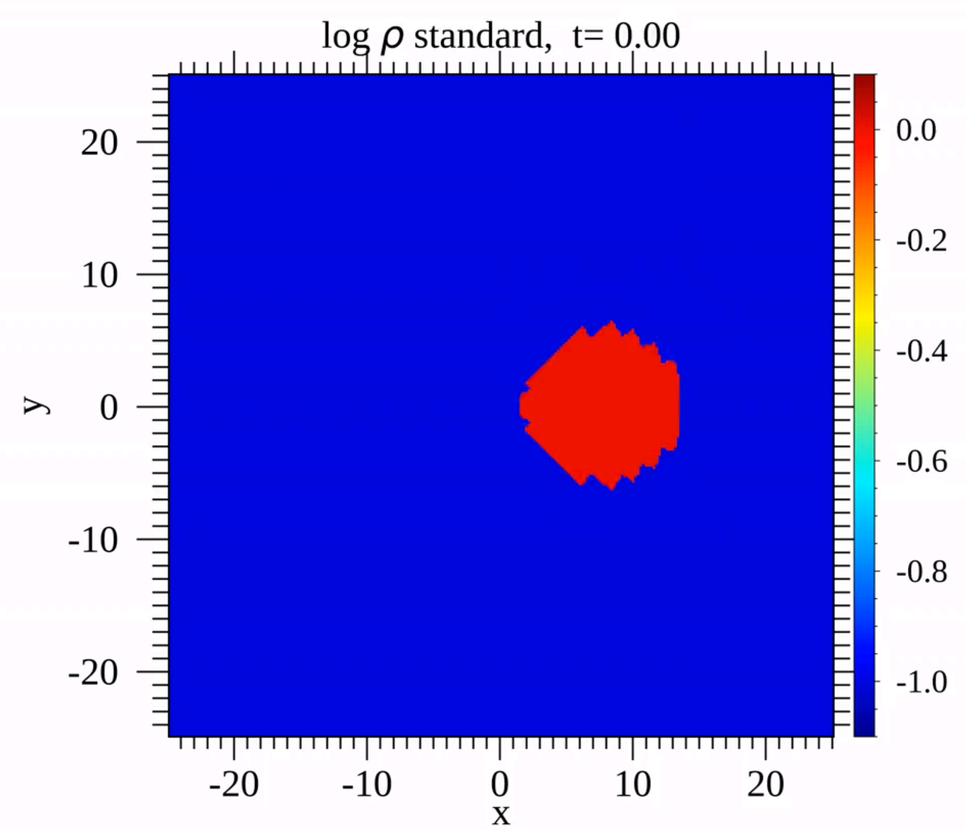
AMR-type variable $\Delta\varphi$



Off-center Explosion

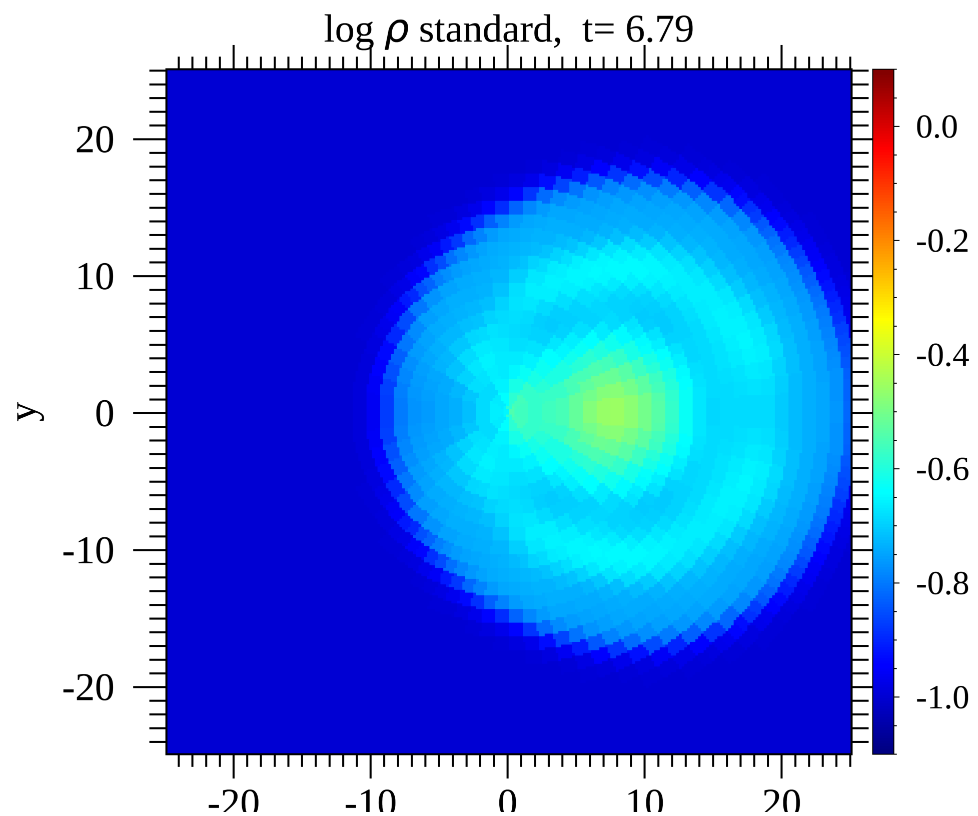
standard high angular resolution→

$= \pi/24$ in $r < 11$

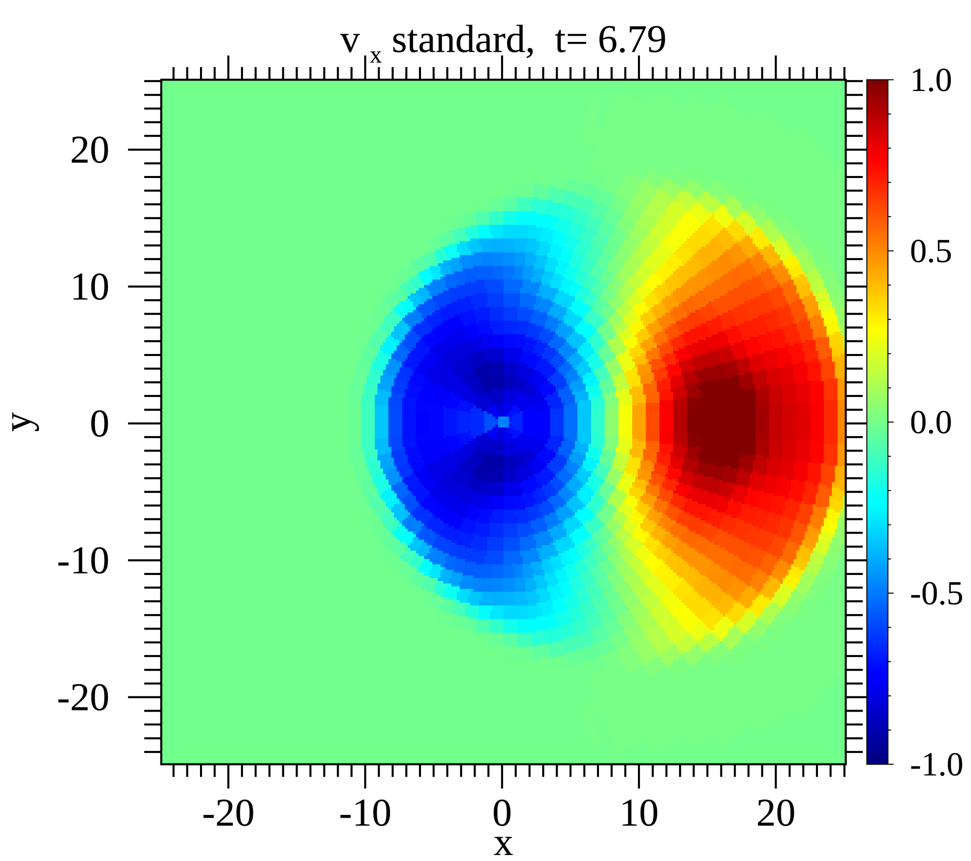


Standard

AMR-type variable $\Delta\varphi$

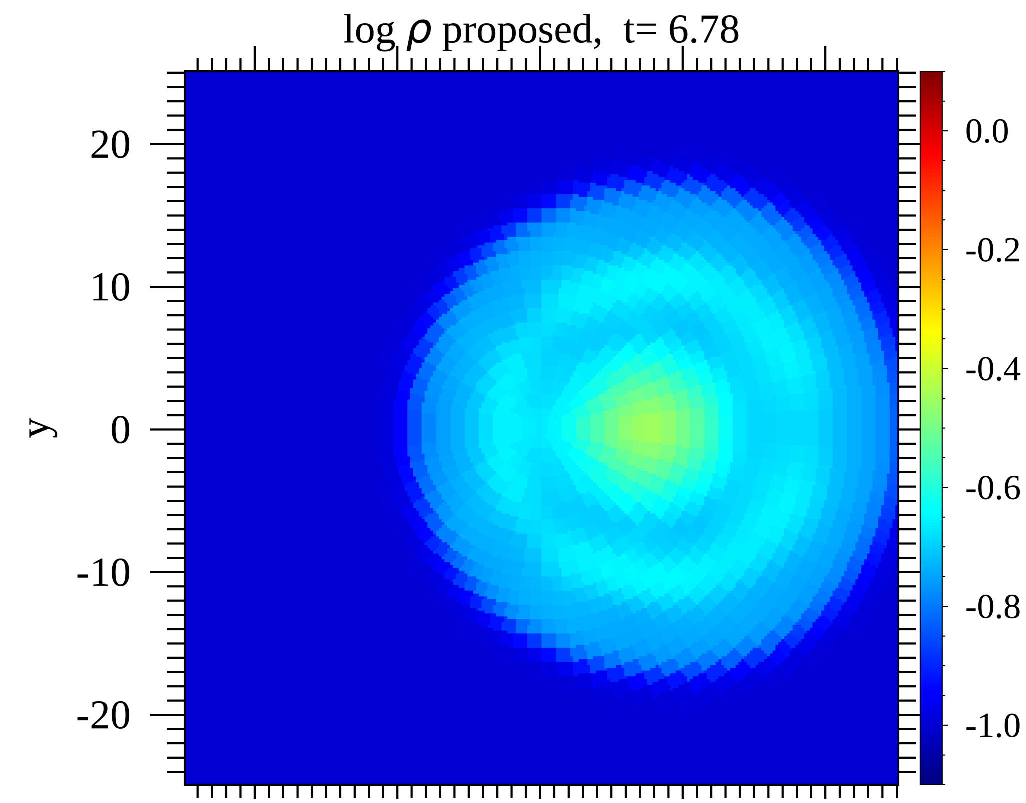


AMR-type variable $\Delta\varphi$

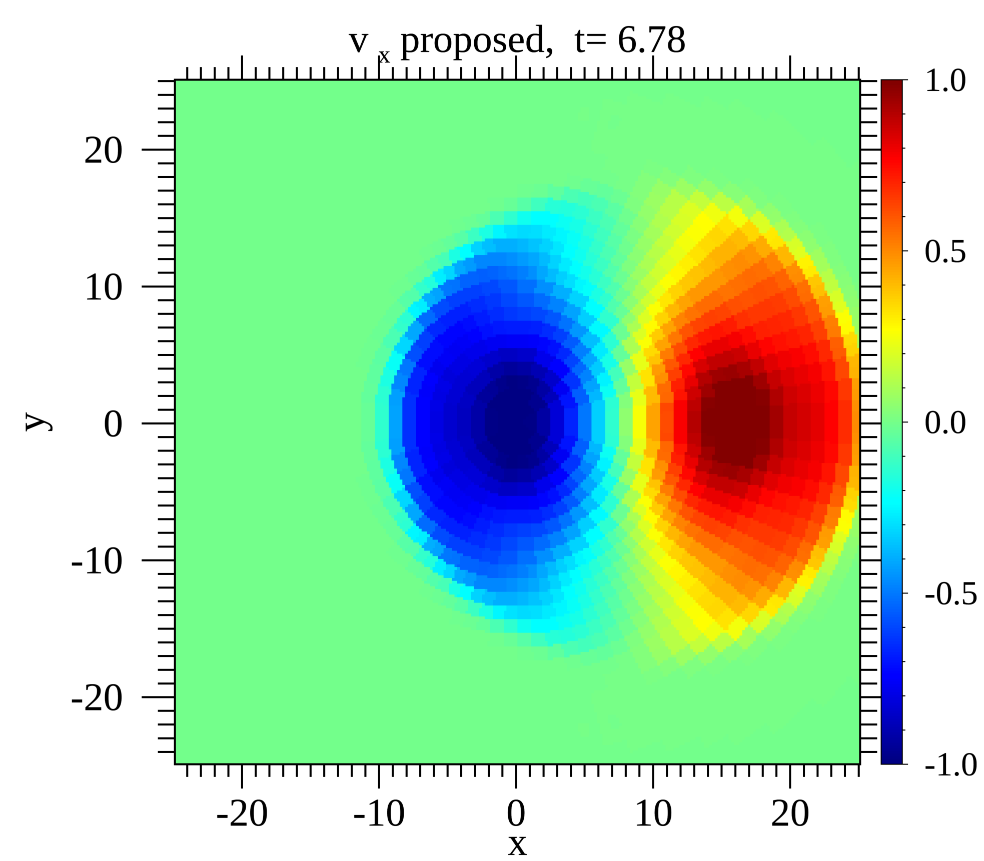


Proposed

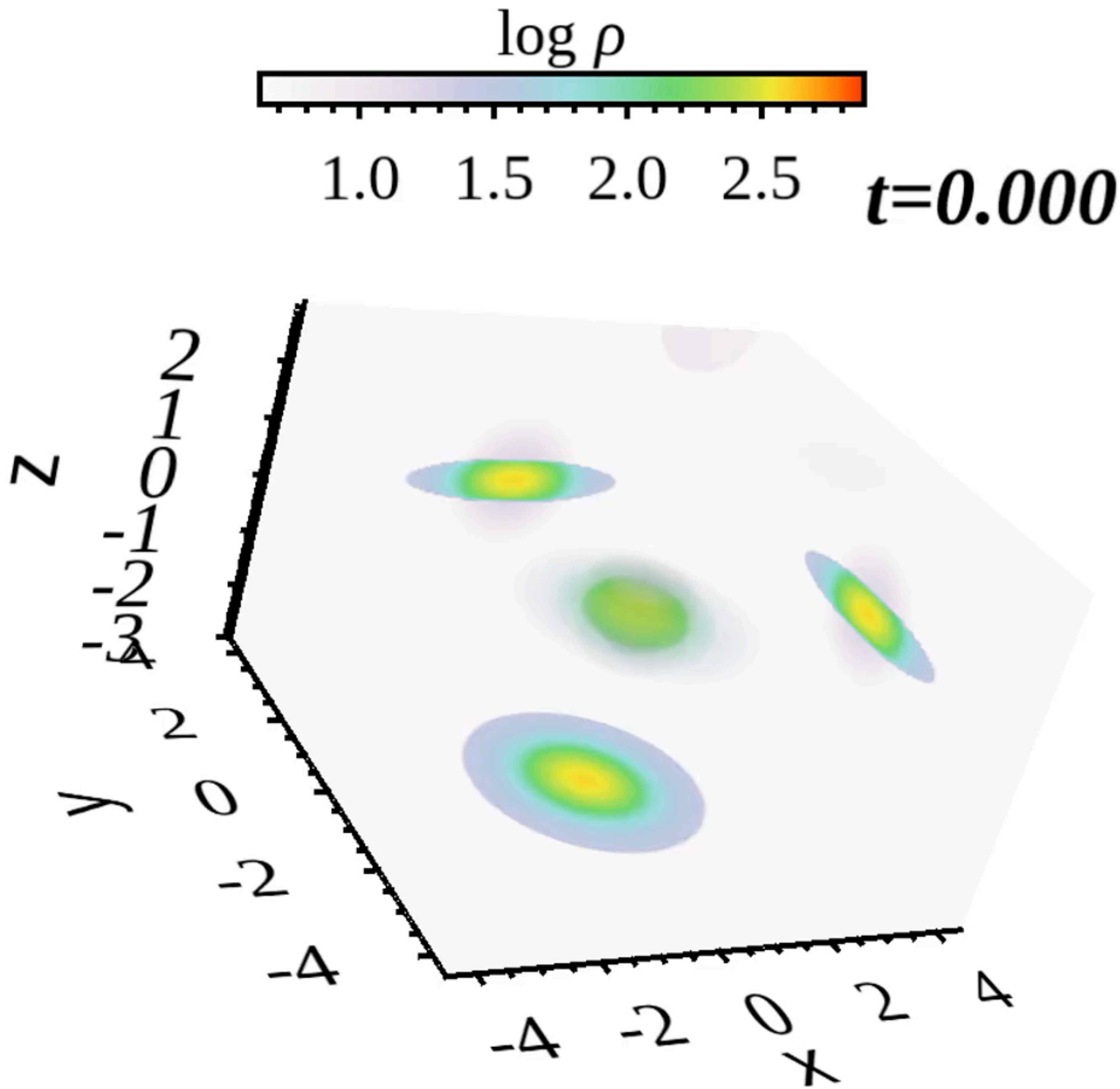
AMR-type variable $\Delta\varphi$



AMR-type variable $\Delta\varphi$



3D test: A gas clump across the *z*-axis



Conclusion

1. We can achieve angular momentum conservation and free-streaming simultaneously.
2. We should use the local Cartesian coordinates oriented on the cell surface when evaluating the numerical flux.
3. We should discriminate the change in the velocity due to a wave from that due to the coordinate.